Switching Diffusions and Applications

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This talk reports some of recent findings involving joint work with

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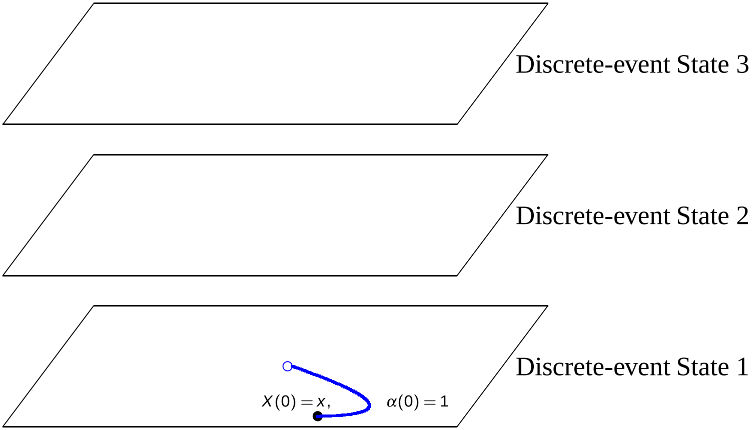
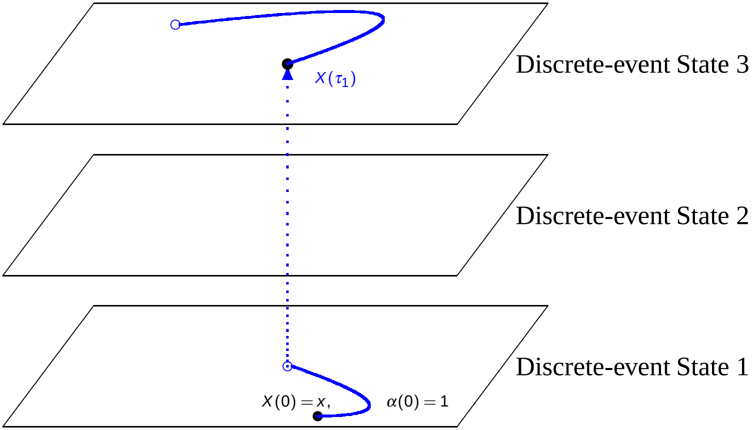
Outline

Hybrid Switching Diffusions: Formulation

Regime-Switching Diffusion: An Illustration



Regime-Switching Diffusion: An Illustration



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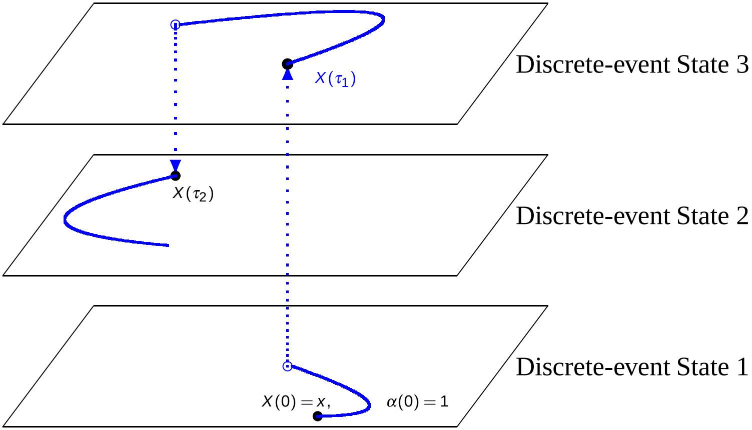

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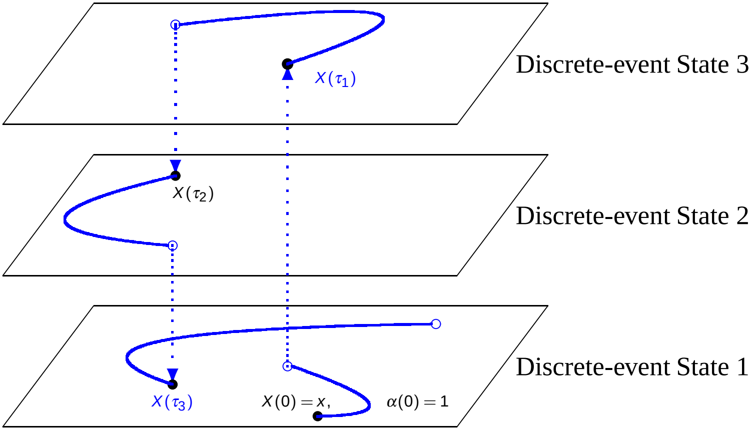

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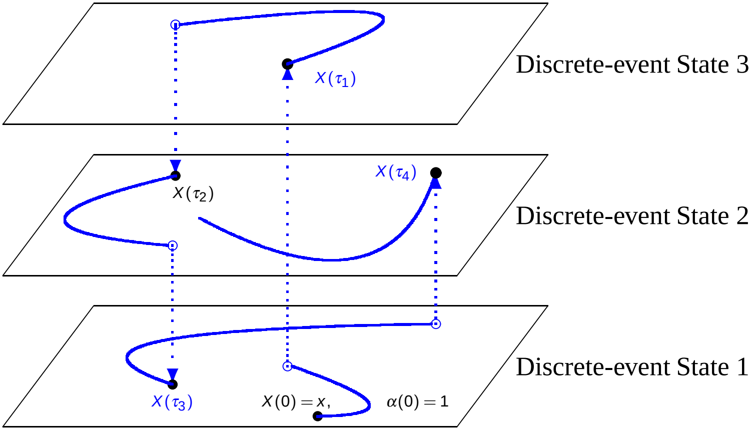

Figure: A “Sample Path” of the Switching Diffusion $(X(t), \alpha(t))$.

Main Features

- continuous dynamics & discrete events coexist
- switching is used to model random environment or other random factors that cannot be formulated by the usual differential equations
- problems naturally arise in applications such as distributed, cooperative, and non-cooperative games, wireless communication, target tracking, reconfigurable sensor deployment, autonomous decision making, learning, etc.
- traditional ODE or SDE models are no longer adequate
- non-Gaussian distribution

Switching Diffusions

$\mathscr{M} = \{1, \dots, m\}$
$\alpha(\cdot)$: taking values in $\mathscr{M}$.
$w(t)$: d-dimensional standard Brownian motion
$b(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$
$\sigma(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r \times \mathbb{R}^d$

$$\begin{aligned} &dX(t) = b(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dw(t), \\ &X(0) = x,\ \alpha(0) = \alpha, \end{aligned} \tag{1}$$

$$P\{\alpha(t+\Delta) = j | \alpha(t) = i, (X(s), \alpha(s)), s \le t\} = q_{ij}(X(t))\Delta + o(\Delta),\ i \neq j. \tag{2}$$

Formulation (cont.)

$Q(x) = (q_{ij}(x))$: generator associated with $\alpha(t)$ satisfying

$$q_{ij}(x) \geq 0, \text{ if } j \neq i, \text{ and } \sum_{j=1}^{m} q_{ij}(x) = 0, \quad i = 1, 2, \ldots, m$$

$\mathscr{L}$: generator of $(X(t), \alpha(t))$. For each $i \in \mathscr{M}$, and any $g(\cdot, i) \in C^2(\mathbb{R}^r)$,

$$\mathscr{L}g(x,i) = \frac{1}{2}\mathrm{tr}(a(x,i)\nabla^2 g(x,i)) + b'(x,i)\nabla g(x,i) + Q(x)g(x,\cdot)(i) \quad (3)$$

where

$\nabla g(\cdot, i)$ & $\nabla^2 g(\cdot, i)$: gradient & Hessian of $g(\cdot, i)$,

$a(x,i) = \sigma(x,i)\sigma'(x,i)$,

$$Q(x)g(x,\cdot)(i) = \sum_{j=1}^{m} q_{ij}(x) g(x,j).$$

Main Difficulty

- Consider $(X(t), \alpha(t))$ with two different initial data $(X(0), \alpha(0)) = (x, \alpha)$ & $(X(0), \alpha(0)) = (y, \alpha)$, $y \neq x$.

- Since $Q(x)$ depends on x, $\alpha^{x,\alpha}(t) \neq \alpha^{y,\alpha}(t)$ infinitely often even though $\alpha^{x,\alpha}(0) = \alpha^{y,\alpha}(0) = \alpha$.

Continuity and Smooth Dependence etc.

Smooth Dependence on Initial Data

Definition

Suppose that $\Psi(x_1, \ldots, x_r, t)$ is a random function. Its partial derivative in mean square with respect to x_i for some $1 \leq i \leq r$ is defined as the random variable $\widetilde{\Psi}(x_1, \ldots, x_r, t)$ such that

$$\mathbf{E}\left| \frac{1}{\Delta x_i}[\Psi(x_1, \ldots, x_i + \Delta x_i, \ldots, x_r, t) - \Psi(x_1, \ldots, x_i, \ldots, x_r, t)] - \widetilde{\Psi}(x_1, \ldots, x_r, t) \right|^2 \to 0 \text{ as } \Delta x_i \to 0.$$

When the mean square partial derivative exists, we normally write it as

$$\widetilde{\Psi}(x_1, \ldots, x_r) = \frac{\partial \Psi(x_1, \ldots, x_r)}{\partial x_i} = \Psi_{x_i}(x_1, \ldots, x_r). \tag{4}$$

Smooth Dependence (cont.)

Theorem

Assume linear growth and global Lipschitz condition. Let $(X^{x,\alpha}(t), \alpha^{x,\alpha}(t))$ be the switching diffusion. Assume that for each $i \in \mathscr{M}$, $b(\cdot, i)$ and $\sigma(\cdot, i)$ have continuous partial derivatives with respect to the variable x up to the second order and that

$$\left|D_x^\beta b(x,i)\right| + \left|D_x^\beta \sigma(x,i)\right| \le K_0(1+|x|^\gamma), \tag{5}$$

where K_0 and γ are positive constants and β is a multi-index with $|\beta| \le 2$. Then $X^{x,\alpha}(t)$ is twice continuously differentiable in mean square with respect to x.

with Zhu, J. Diff. Eqs. (2010)

Associated Poisson Measure

- $\Delta_{ij}(x)$: left closed, right open intervals of $\mathbb{R}$, with length $q_{ij}(x)$
- $h : \mathbb{R}^r \times \mathscr{M} \times \mathbb{R} \mapsto \mathbb{R}$:

$$h(x,i,z) = \sum_{j=1}^{m} (j-i) I\{z \in \Delta_{ij}(x)\}. \tag{6}$$

$$d\alpha(t) = \int_{\mathbb{R}} h(X(t), \alpha(t-), z) \mathfrak{p}(dt, dz), \tag{7}$$

where
$\mathfrak{p}(dt,dz)$: a Poisson random measure with intensity $dt \times m(dz)$,
m: the Lebesgue measure on $\mathbb{R}$,
$\mathfrak{p}(\cdot,\cdot)$ independent of $w(\cdot)$.

Generalized Itô Lemma

If $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^r \times \mathscr{M})$, then for any $t \geq 0$:

$$\begin{aligned} V(t,X(t),\alpha(t)) =& V(0,X(0),\alpha(0)) \\ &+ \int_0^t \left[\frac{\partial}{\partial s} + \mathscr{L}\right] V(s,X(s),\alpha(s))ds + M_1(t) + M_2(t), \end{aligned} \tag{8}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \langle \nabla V(s,X(s),\alpha(s)), \sigma(X(s),\alpha(s))dw(s)\rangle. \\ M_2(t) &= \int_0^t \int_{\mathbb{R}} \big[V(s,X(s),\alpha(0)+h(X(s),\alpha(s-),z)) \\ &\qquad\qquad - V(s,X(s),\alpha(s))\big]\mu(ds,dz), \end{aligned}$$

$\mu(ds,dz) = \mathfrak{p}(ds,dz) - ds \times m(dz)$ is a martingale measure.

The Strong Feller Property

Definition

$(X(t), \alpha(t))$ is *strong Feller* if $(x, i) \mapsto \mathbf{E}_{x,i} f(X(t), \alpha(t))$ is continuous for any $f \in \mathscr{B}_b(\mathbb{R}^n \times \mathscr{M})$.

$\mathbf{P}_{x,i}$: the probability law of $(X^{x,i}(t), \alpha^{x,i}(t))$,

$\mathbf{E}_{x,i}$: the corresponding expectation.

Literature

- Feller, Dynkin, Girsanov: diffusions
- Fattler and Grothaus, 2007: strong Feller properties for distorted Brownian motion with reflecting boundary condition,
- Jaśkiewicz and Nowak, 2006: zero-sum ergodic stochastic games with Feller transition probabilities,
- Peszat and Zabczyk, 1995: strong Feller property for diffusions on Hilbert spaces, equations driven by Lévy processes,
- Szarek, 2006: Feller processes on nonlocally compact spaces, and
- Taira et al., 2001: Feller semigroups and degenerate elliptic operators with Wentzell boundary conditions

- Not much work for regime switching diffusions.

Assumptions

(A1) For $i = 1, 2, \ldots, m$ and $j, k = 1, 2, \ldots, n$, the coefficients $b_j(x, i)$, $\sigma_{jk}(x, i)$, and $q_{ij}(x)$ are Hölder continuous with exponent $0 < \gamma \leq 1$.

(A2) $Q(x)$ is irreducible for each $x \in \mathbb{R}^n$.

(A3) For all $(x, i) \in \mathbb{R}^n \times \mathscr{M}$, $a(x, i) = (a_{jk}(x, i))$ is symmetric and satisfies

$$\langle a(x,i)\xi, \xi \rangle \geq \kappa |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{9}$$

where $\kappa > 0$ is some constant.

The Strong Feller Property (I)

Lemma

Assume in addition to (A1)–(A3) *that for* $i, \ell = 1, 2, \ldots, m$ *and* $j, k = 1, 2, \ldots, n$*, the coefficients* $a_{jk}(x, i)$*,* $b_j(x, i)$*, and* $q_{i\ell}(x)$ *are bounded. Then the process* $(X(t), \alpha(t))$ *is strong Feller.*

Sketch of Proof

1. By [Eidelman, 1969, Themrem 2.1], $\dfrac{\partial u}{\partial t} = \mathscr{L} u$ has a unique fundamental soln. $p(x,i,t,y,j)$, which is positive and satisfies

$$\left|D_x^{\theta} p(x,i,t,y,j)\right| \le C t^{\frac{-n+|\theta|}{2}} \exp\left\{ \frac{-c\,|y-x|^2}{t} \right\}. \tag{10}$$

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2. For any $\phi(x,i) \in C_b(\mathbb{R}^n \times \mathscr{M})$, define

$$\Phi(t,x,i) := \sum_{j=1}^m \int_{\mathbb{R}^n} p(x,i,t,y,j)\phi(y,j)dy.$$

Then it can be shown (stoch. representation) that $\Phi(t,x,i) = \mathbf{E}_{x,i}\phi(X(t),\alpha(t))$. So p is the transition probability density of $(X^{x,i}(t),\alpha^{x,i}(t))$.

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3. Finally for any $f(x,i) \in \mathscr{B}_b(\mathbb{R}^n \times \mathscr{M})$,

$$x \mapsto \mathbf{E}_{x,i} f(X(t),\alpha(t)) = \sum_{j=1}^{m} \int_{\mathbb{R}^n} f(y,j)p(x,i,t,y,j)dy$$

is continuous by the Dominated Convergence Theorem.

The Strong Feller Property II

Theorem

Assume (A1)–(A3) *hold. Then the process* $(X(t), \alpha(t))$ *possesses the strong Feller property.*

The Strong Markov Property

To summarise, under conditions (A1)–(A3), the process $(X(t), \alpha(t))$ is

- càdlàg (sample paths are right continuous with left limits); and
- strong Feller.

So it is *strong Markov*

Seemingly Not Much Different from Diffusions without Switching?

Q: When we have a coupled system with $\mathscr{M} = \{1,2\}$ and two stable linear systems, do we always get a stable system?

Consider $\dot{x} = A(\alpha(t))x + B(\alpha(t))u(t)$, and a state feedback $u(t) = K(\alpha(t))x(t)$. Then one gets

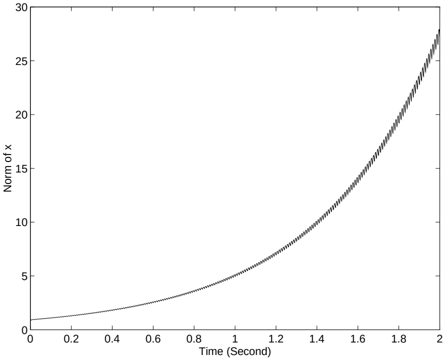
$$\dot{x} = [A(\alpha(t)) - B(\alpha(t))K(\alpha(t))]x.$$

Suppose that $\alpha(t) \in \{1,2\}$ such that

$$A(1) - B(1)K(1) = \begin{bmatrix} -100 & 20 \\ 200 & -100 \end{bmatrix}, \quad A(2) - B(2)K(2) = \begin{bmatrix} -100 & 200 \\ 20 & -100 \end{bmatrix}.$$

The two feedback systems are stable individually. But if we choose $\alpha(t)$ so that it switches at $k\eta$, where $\eta = 0.01$. Then the resulting system is unstable.

The hybrid system is unstable (curtesy of Le Yi Wang)



[L.Y. Wang, P.P. Khargonecker, and A. Beydoun, 1999]

Why is the system unstable?

$$\frac{1}{2}[A(1)-B(1)K(1)+A(2)-B(2)K(2)]=\frac{1}{2}\begin{bmatrix} -200 & 220 \\ 220 & -200 \end{bmatrix}$$

is an unstable matrix.

The averaging effect dominates the dynamics.

An Example

Consider

$$\dot{x}(t) = A(\alpha(t))x(t) \tag{11}$$

where $\alpha(t)$ has two states $\{1, 2\}$,

$$A(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \qquad Q = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix},$$

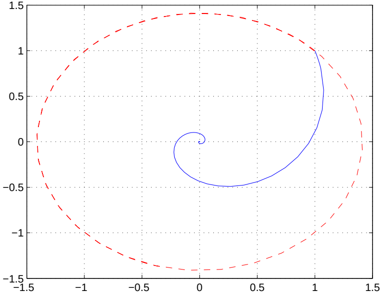
Associated with the hybrid system, there are two ODEs

$$\dot{x}(t) = A(1)x(t), \quad \text{and} \tag{12}$$

$$\dot{x}(t) = A(2)x(t) \tag{13}$$

switching back and forth according to $\alpha(t)$.

Phase Portrait of the Components



Phase portraits of the ‘component’ with a center (in dashed line) and the ‘component’ with a stable node (in solid line) with the same initial condition $x_0 = [1, 1]'$

Phase Portrait of Hybrid System

The phase portrait is given below.

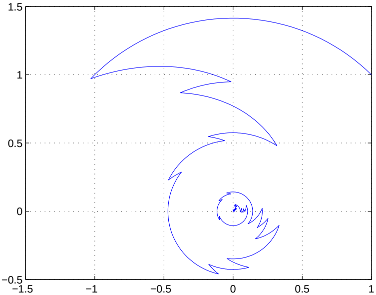


Figure: Switching linear system: Phase portrait of (11) with $x_0 = [1, 1]'$.

with Zhu and Song, (2009), Quart. Appl. Math

Examples

Insurance Risk Models

The surplus at time t:

$$S(t,x,i) = x + \int_0^t c(\alpha(s))ds - \sum_{j=1}^{N(t)} X_j(\alpha(T_j)),$$

- (x,i): initial (surplus, regime);
- $c(i)$: premium rate;
- $X_j(i)$: claim size;
- T_j: claim time;
- $N(t)$: Poisson process.
- $\alpha(t)$ is used to model:
  - El Nino/La Nina phenomena in property ins.
  - economic condition in unemployment policy
  - certain epidemics in health insurance

with Y. Liu & H.L. Yang, *Scandinavian Actuarial J.* (2006)

- Regime-switching market models

$$dS(t) = \mu(\alpha(t))S(t)dt + \sigma(\alpha(t))S(t)dw.$$

  - both the return rate & volatility depend on $\alpha(t)$
  - $\alpha(\cdot)$ and $w(\cdot)$ are independent
  - $\alpha(t)$: market mode, investor's mode, & other economic factors (e.g., bull, bear)

with X.Y. Zhou, *SIAM J. Control Optim.* (2003), IEEE T-AC, (2004)

Average Cost Per Unit Time Problem

Consider a controlled switching diffusion $(X(t), \alpha(t))$ (drift and diffusion coefficients also depend on a control u).

Aim: find $u^*(\cdot)$ so

$$\lim_{T\to\infty} E \frac{1}{T} \int_0^T L(X(t), \alpha(t), u(t))dt$$

is minimized.

Questions: Does there exist an ergodic measure? If yes, can we replace the instantaneous measure by the ergodic one?

Two-time-scale Markov Chains

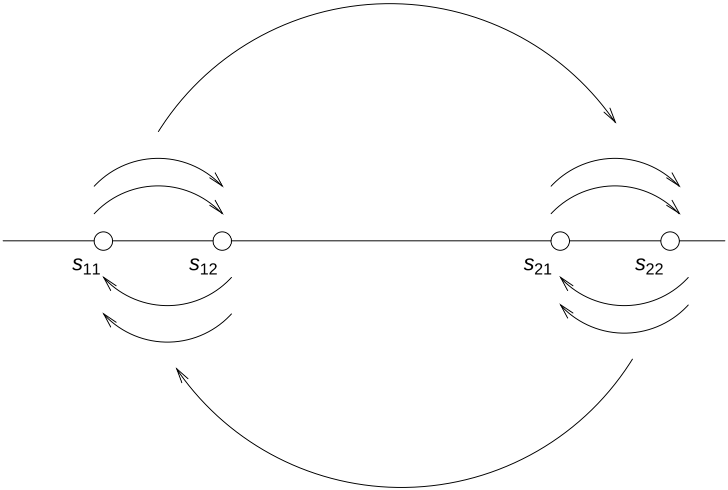
- Two-time-scale Markov chains $\varepsilon > 0$ small,

$$Q(t) = Q^{\varepsilon}(t) = \frac{\widetilde{Q}(t)}{\varepsilon} + \widehat{Q}(t). \tag{14}$$

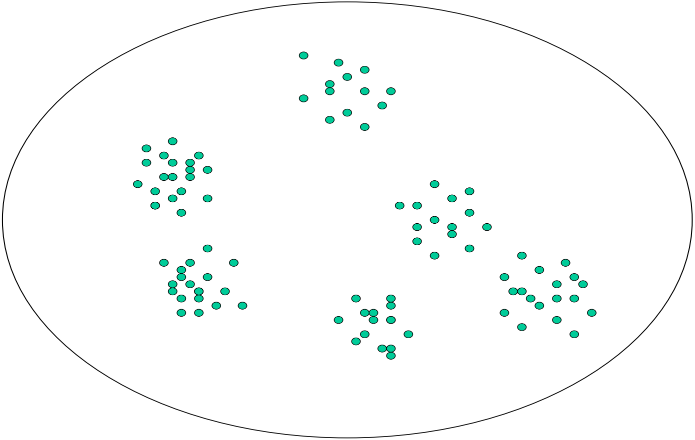
  - $\widetilde{Q}(t)$, $\widehat{Q}(t)$ are generators of Markov chains.
  - $\widetilde{Q}(t) = \mathrm{diag}(\widetilde{Q}^1(t), \dots, \widetilde{Q}^l(t))$ nearly decomposable
  - $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l$; $\mathcal{M}_i = \{s_{i1}, \dots, s_{im_i}\}$

work with Q. Zhang book (1998), Ann Appl. Probab. (1996, 2000, 2007 H.Q. Zhang) etc.

Two-time Scale (a demonstration)



Limit of $\dfrac{1}{\sqrt{\varepsilon}}\int_0^t [I_{\{\alpha^\varepsilon(s)=s_{ij}\}} - \nu_j^i(s) I_{\{\alpha^\varepsilon(s)\in\mathcal{M}_i\}}]ds$



Mean-Field Model

- $\alpha(t)$: with $\mathcal{M} = \{1, 2, \dots, m_0\}$.
- Consider an ℓ-body mean-field model For $i = 1, 2, \dots, \ell$,

$$\begin{aligned} dX_i(t) = & \left[\gamma(\alpha(t))X_i(t) - X_i^3(t) - \beta(\alpha(t))(X_i(t) - \overline{X}(t))\right]dt \\ & + \sigma_{ii}(X(t), \alpha(t))dw_i(t), \\ \overline{X}(t) = & \frac{1}{\ell}\sum_{j=1}^{\ell} X_j(t), \\ X(t) = & (X_1(t), X_2(t), \dots, X_\ell(t))', \end{aligned} \tag{15}$$

  $\gamma(i) > 0$ and $\beta(i) > 0$ for $i \in \mathcal{M}$.
- Originated from statistical mechanics, mean-field models are concerned with many-body systems with interactions. To overcome the difficulty of interactions due to the many bodies, one of the main ideas is to replace all interactions to any one body with an average or effective interaction.

Consensus Problems: Schooling (Couzin [Nature, 2005])

Consensus Problems: High Way Traffic (curtesy of Mingyi Huang)

Recurrence, Ergodicity, Stability

Regularity & Recurrence

Definition

Regularity. A Markov process $Y^{x,\alpha}(t) = (X^{x,\alpha}(t), \alpha^{x,\alpha}(t))$ is said to be *regular*, if for any $0 < T < \infty$,

$$\mathbf{P}\{\sup_{0\le t\le T} |X^{x,\alpha}(t)| = \infty\} = 0. \tag{16}$$

Remark

Let $\beta_n := \inf\{t : |X^{x,\alpha}(t)| = n\}$. Then $\{\beta_n\}$ is monotonically increasing and hence has a (finite or infinite) limit. It follows that the process is regular iff

$$\beta_n \to \infty \text{ almost surely as } n \to \infty. \tag{17}$$

Definition

(i) *Recurrence.* For $U := D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^r$ is an open set with compact closure, let $\sigma_U^{x,\alpha} = \inf\{t : Y^{x,\alpha}(t) \in U\}$. A regular process $Y^{x,\alpha}(\cdot)$ is *recurrent* w.r.t. U if

$$\mathbf{P}\{\sigma_U^{x,\alpha} < \infty\} = 1 \;\text{ for any } (x,\alpha) \in D^c \times \mathcal{M}.$$

(ii) *Positive and Null Recurrence.* A recurrent process satisfying $\mathbf{E}\sigma_U^{x,\alpha} < \infty$ is said to be *positive recurrent* w.r.t. U; otherwise, the process is *null recurrent* w.r.t. U.

Recurrence Is Independent of Sets

(i) The process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. $D \times \mathscr{M}$ if and only if it is (positive) recurrent w.r.t. $D \times \{\ell\}$, where $D \subset \mathbb{R}^r$ is a bounded open set with compact closure and $\ell \in \mathscr{M}$.

(ii) If the process $(X(t), \alpha(t))$ is (positive) recurrent w.r.t. some $U = D \times \mathscr{M}$, where $D \subset \mathbb{R}^r$, then it is (positive) recurrent w.r.t. $\widetilde{U} = \widetilde{D} \times \mathscr{M}$, where $\widetilde{D} \subset \mathbb{R}^r$ is any nonempty open set.

Positive Recurrence

Theorem

A necessary and sufficient condition for positive recurrence with respect to a domain $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathscr{M}$ is: For each $i \in \mathscr{M}$, there exists a nonnegative function $V(\cdot, i) : D^c \mapsto \mathbb{R}$ s.t. $V(\cdot, i)$ is twice continuously differentiable and that

$$\mathscr{L}V(x,i) = -1, \ \ (x,i) \in D^c \times \mathscr{M}. \tag{18}$$

Let $u(x,i) = \mathbf{E}_{x,i}\sigma_D$. It is the smallest positive sol'n to

$$\begin{cases} \mathscr{L}u(x,i) = -1, & (x,i) \in D^c \times \mathscr{M}, \\ u(x,i) = 0, & (x,i) \in \partial D \times \mathscr{M}. \end{cases} \tag{19}$$

Step 1: Positive recurrence. Show the process is positive recurrent if *exists* $V(\cdot,\cdot)$ (≥ 0) satisfying the conditions of the theorem.

- Fix any $(x,i) \in D^c \times \mathscr{M}$ and set $\sigma_D^{(n)}(t) = \min\{\sigma_D, t, \beta_n\}$. Dynkin's formula implies

$$\begin{aligned} \mathbf{E}_{x,i} V(X(\sigma_D^{(n)}(t)), \alpha(\sigma_D^{(n)}(t))) - V(x,i) &= \mathbf{E}_{x,i} \int_0^{\sigma_D^{(n)}(t)} \mathscr{L} V(X(s), \alpha(s)) ds \\ &= -\mathbf{E}_{x,i} \sigma_D^{(n)}(t). \end{aligned}$$

- Since $V(\cdot)$ is nonnegative,

$$\mathbf{E}_{x,i} \sigma_D^{(n)}(t) \leq V(x,i).$$

- Letting $n \to \infty$ and $t \to \infty$, $\mathbf{E}_{x,i} \sigma_D < \infty$. This is positive recurrence.

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- Letting $n \to \infty$ and $t \to \infty$, $\mathbf{E}_{x,i} \sigma_D < \infty$. This is positive recurrence.

Step 1: Positive recurrence. Show the process is positive recurrent if *exists* $V(\cdot,\cdot)$ (≥ 0) satisfying the conditions of the theorem.

- Fix any $(x,i) \in D^c \times \mathscr{M}$ and set $\sigma_D^{(n)}(t) = \min\{\sigma_D, t, \beta_n\}$. Dynkin's formula implies

$$\begin{aligned}\mathbf{E}_{x,i} V(X(\sigma_D^{(n)}(t)), \alpha(\sigma_D^{(n)}(t))) - V(x,i) &= \mathbf{E}_{x,i} \int_0^{\sigma_D^{(n)}(t)} \mathscr{L} V(X(s), \alpha(s)) ds \\ &= -\mathbf{E}_{x,i} \sigma_D^{(n)}(t).\end{aligned}$$

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Step 2: Show $u(x,i) := \mathbf{E}_{x,i}\sigma_D < \infty$ is the smallest positive solution of the BVP (19).

- Set $\sigma_D^{(n)} = \min\{\sigma_D, \beta_n\}$ & $u_n(x,i) = \mathbf{E}_{x,i}\sigma_D^{(n)}$. Then $u_n(x,i)$ solves

$$\mathscr{L}u_n(x,i) = -1,\ u_n(x,i)|_{x\in\partial D} = 0\ u_n(x,i)|_{|x|=n} = 0.$$

- $v_n(x,i) := u_{n+1}(x,i) - u_n(x,i)$ is $\mathscr{L}$-harmonic in $(D^c \cap \{|x| < n\}) \times \mathscr{M}$.
- $\mathbf{E}_{x,i}\sigma_D^{(n)} \nearrow \mathbf{E}_{x,i}\sigma_D$ by regularity and DCT. Hence we can write

$$u(x,i) = u_{n_0}(x,i) + \sum_{k=n_0}^{\infty} v_k(x,i).$$

- Harnack's theorem implies that $u(x,i)$ is a solution of (19).
- Maximum Principle yields $u(x,i)$ is the smallest solution.

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Positive Recurrence (2)

A necessary & sufficient condition for positive recurrence w.r.t. $U = D \times \{\ell\} \subset \mathbb{R}^r \times \mathscr{M}$ is: For each $i \in \mathscr{M}$, there exists a nonnegative function $V(\cdot, i) : D^c \mapsto \mathbb{R}$ s.t. $V(\cdot, i)$ is twice continuously differentiable and that for some $\gamma > 0$,

$$\mathscr{L}V(x,i) \le -\gamma, \quad (x,i) \in D^c \times \mathscr{M}.$$

Ergodicity

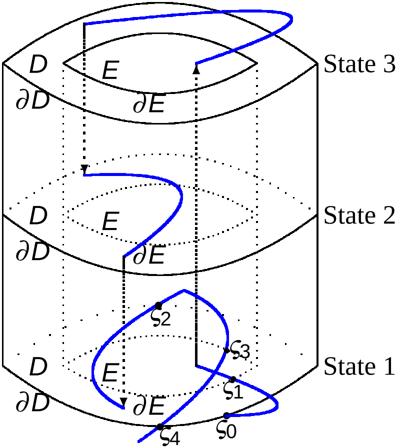

Figure 2: Cycles of $Y(t) = (X(t), \alpha(t))$; $m = 3$ & $\ell = 1$

Cycles

- Assume the process is positive recurrent w.r.t. $U = E \times \{\ell\}$; $E \subset \mathbb{R}^r$ and $\ell \in \mathscr{M}$ are fixed from now on.
- Let ∂E be sufficiently smooth. Let $D \subset \mathbb{R}^r$ be a bdd. ball with suff. smooth ∂D s.t. $E \cup \partial E \subset D$.
- Let $\varsigma_0 = 0$ and then define for $n = 0, 1, \dots$

$$\begin{aligned} \varsigma_{2n+1} &= \inf\{t \geq \varsigma_{2n} : (X(t), \alpha(t)) \in \partial E \times \{\ell\}\}, \\ \varsigma_{2n+2} &= \inf\{t \geq \varsigma_{2n+1} : (X(t), \alpha(t)) \in \partial D \times \{\ell\}\}. \end{aligned}$$

Then we can divide an arbitrary sample path of the process into cycles:

$$[\varsigma_0, \varsigma_2), [\varsigma_2, \varsigma_4), \dots, [\varsigma_{2n}, \varsigma_{2n+2}) \dots. \tag{20}$$

- Assume $Y(0) = (X(0), \alpha(0)) = (x, \ell) \in \partial D \times \{\ell\}$.
- Define $Y_n = Y(\varsigma_{2n}) = (X_n, \ell), n = 0, 1, \ldots$ It is a MC on $\partial D \times \{\ell\}$ by strong Markov property

Theorem

A positive recurrent process $(X(t), \alpha(t))$ has a unique stationary distribution $\widehat{\nu}(\cdot, \cdot) = (\widehat{\nu}(\cdot, i) : i \in \mathscr{M})$.

Strong Law of Large Numbers

Theorem

Denote by $\mu(\cdot,\cdot)$ *the stationary density associated with* $\widehat{v}(\cdot,\cdot)$ *and* $f(\cdot,\cdot) : \mathbb{R}^r \times \mathcal{M} \mapsto \mathbb{R}$ *is Borel measurable such that*

$$\sum_{i=1}^{m_0} \int_{\mathbb{R}^r} |f(x,i)|\mu(x,i)dx < \infty. \tag{21}$$

Then for any $(x,i) \in \mathbb{R}^r \times \mathcal{M}$

$$\mathbf{P}_{x,i}\left(\frac{1}{T}\int_0^T f(X(t),\alpha(t))dt \to \overline{f} \right) = 1, \tag{22}$$

where $\overline{f} = \sum_{i=1}^{m_0} \int_{\mathbb{R}^r} f(x,i)\mu(x,i)dx.$

Cauchy Problem

Let the assumptions of the last theorem be satisfied, and $u(t,x,i)$ be the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,x,i)}{\partial t} &= \mathscr{L}u(x,i),\ i \in \mathscr{M}, \\ u(0,x,i) &= f(x,i). \end{cases} \tag{23}$$

Then as $T \to \infty$,

$$\frac{1}{T}\int_0^T u(t,x,i)dt \to \sum_{i=1}^{m_0}\int_{\mathbb{R}^r} f(x,i)\mu(x,i)dx. \tag{24}$$

A key to establish this result is the result of law of large numbers.

Explosion Suppression & Stabilization

Regularity Criterion (cont.)

Theorem

Suppose that $b(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^r$ *and that* $\sigma(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^{r\times d}$*,*

$$\begin{array}{l} dX(t)=b(X(t),\alpha(t))dt+\sigma(X(t),\alpha(t))dw(t),\ (X(0),\alpha(0))=(x,\alpha), \\ P\{\alpha(t+\delta)=j|\alpha(t)=i,X(s),\alpha(s),s\le t\}=q_{ij}(X(t))\delta+o(\delta),\ i\neq j. \end{array} \tag{25}$$

Suppose that for each $i\in\mathcal{M}$*, both* $b(\cdot,i)$ *and* $\sigma(\cdot,i)$ *are local linear growth and local Lipschitzian and that* $\exists$ *a nonnegative* $V(\cdot,\cdot):\mathbb{R}^r\times\mathcal{M}\mapsto\mathbb{R}^+$ *that is* C^2 *in* $x\in\mathbb{R}^r$ *for each* $i\in\mathcal{M}$ s.t. $\exists\gamma_0>0$

$$\begin{array}{l} \mathcal{L}V(x,i)\le\gamma_0V(x,i),\ \textit{for all}\ (x,i)\in\mathbb{R}^r\times\mathcal{M}, \\ V_R:=\displaystyle\inf_{|x|\ge R,\ i\in\mathcal{M}}V(x,i)\to\infty\ \ \textit{as}\ \ R\to\infty. \end{array} \tag{26}$$

Then the process $(X(t),\alpha(t))$ *is regular.*

Explosion Suppression

$x \in \mathbb{R}^r$

$f(\cdot,\cdot) : \mathbb{R}^r \times \mathscr{M} \mapsto \mathbb{R}^r$

$\alpha(t) \in \mathscr{M} = \{1, \ldots, m\}$

$$\frac{dX(t)}{dt} = f(X(t), \alpha(t)) \tag{27}$$

$f(\cdot, i)$ continuous but the growth rate is faster than linear

We wish to stabilize (27).

Motivational Example

- Consider an even simpler problem: the logistic system

$$\dot{x}(t) = x(t)(1 + x(t)),\ x(0) = 1.$$

- solution:

$$x(t) = \frac{1}{-1 + 2e^{-t}}.$$

- It will blow up and the explosion time $\tau = \log 2$.

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- Question: How can we get a global soln; how can we stabilize this?

  Two things are needed:

  1) extend to a global solution;
  2) stabilization.

What have been done?

- Khasminskii's book (1981): stabilization using white noise
- Arnold (1972): $\dot{x} = Ax$ can can be stabilized by zero mean stationary process iff $\mathrm{tr}(A) < 0$
- Mao (1994) established a general stabilization results of Brownian noise under linear growth condition.
- Wu & Hu (2009) treated one-sided growth condition
- Mao, Yin, and Yuan (2007): showed that both Brownian motion and Markov Chain can be used to stabilize systems.

Motivation (diffusion case)

$$dx = \mu x dt + \sigma x dw, \; x(0) = x_0.$$

$$x(t) = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma w(t)\right).$$

$$\text{when} \;\; \sigma^2 > 2\mu,$$

$$\limsup_t \frac{\log|x(t)|}{t} \leq \left(\mu - \frac{\sigma^2}{2}\right) < 0.$$

This implies exponential stability.

How to Get a Global Solution? Stablization?

- add a diffusion perturbation

$$dX(t) = f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^\beta X(t)dw_1(t)$$

  such that $2\beta - \beta_1 > 0$, where $w_1(\cdot)$ is scalar Brownian motion.
- add another diffusion to get stability

$$\begin{aligned} dX(t) &= f(X(t), \alpha(t))dt + a_1(\alpha(t))|X(t)|^\beta X(t)dw_1(t) \\ &\quad + a_2(\alpha(t))X(t)dw_2(t), \end{aligned} \tag{28}$$

  where $w_2(\cdot)$ is a scalar Brownian motion independent of $w_1(\cdot)$.
- More general,

$$dX(t) = f(X(t), \alpha(t))dt + \sigma_1(X(t), \alpha(t))dw_1 + \sigma_2(X(t), \alpha(t))dw_2. \tag{29}$$

Conditions

(a) $f(0,i) = \sigma_1(0,i) = \sigma_2(0,i) = 0$;

(b) $f'(x,i)x \leq K_0(|x|^{\beta_1+2} + |x|^2)$ for each $i \in \mathscr{M}$ and some $\beta_1 > 0$.

(c) for some $\beta > 0$ satisfying $2\beta - \beta_1 > 0$ and some $K_j > 0$ with $j = 1, \ldots, 4$ satisfying $2K_1 > K_2$ and for each $x \in \mathbb{R}^r$,

$$\begin{aligned}
&\mathrm{tr}(\sigma_1(x,i)\sigma_1'(x,i)xx') \geq K_1(|x|^{4+2\beta} - |x|^4) \\
&\mathrm{tr}(\sigma_1(x,i)\sigma_1'(x,i)) \leq K_2(|x|^{2+2\beta} + |x|^2), \\
&\mathrm{tr}(\sigma_2(x,i)\sigma_2'(x,i)xx') \geq K_3|x|^4, \\
&\mathrm{tr}(\sigma_2(x,i)\sigma_2'(x,i)) \leq K_4|x|^2.
\end{aligned} \tag{30}$$

Results

- We can get a global solution
- $\limsup_{t\to\infty} P(|X(t)| \geq K_\delta) \leq \delta$
- The resulting system is stable w.p.1. In fact, $\limsup_t \log|X(t)|/t < 0$ w.p.1.

Discrete Approximation

$$x_{n+1} = x_n + \mu f(x_n, \alpha_n) + \sqrt{\mu}\sigma(\alpha_n)|x_n|^{\beta} x_n \eta_n + \sqrt{\mu}\ell(\alpha_n) x_n \xi_n. \qquad (31)$$

Define

$$x^{\mu}(t) = x_n, \ \alpha^{\mu}(t) = \alpha_n, \ \ t \in [n\mu, n\mu + \mu).$$

Theorem

Under suitable assumptions, $(x^{\mu}(\cdot), \alpha^{\mu}(\cdot))$ *converges weakly to* $(x(\cdot), \alpha(\cdot))$ *such that the limit is the solution to the martingale problem with operator* $\mathscr{L}$*. That is,* $(x(\cdot), \alpha(\cdot))$ *is the solution of SDE with switching.*

Theorem

Under assumptions of Theorem 12*,* (x_n, α_n) *is regular. That is, the process* (x_n, α_n) *does not blow up in finite time.*

Theorem

Under the conditions of Theorem 13*, for any* $\delta > 0$ *sufficiently small, there is a* $K_\delta > 0$ *such that*

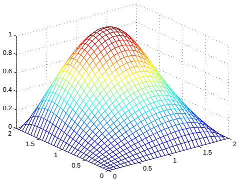
$$\limsup_{n\to\infty} P(|x_n| \geq K_\delta) \leq \delta. \tag{32}$$

Under suitable conditions, for any $t_\mu \to \infty$ as $\mu \to 0$,

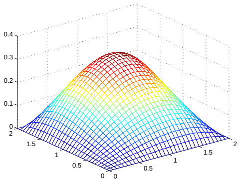
$$\limsup_{\mu \to 0} \frac{\log |x^\mu(t_\mu)|}{t_\mu} \le 0 \ \text{ w.p.1}, \tag{33}$$

where the probability measure is understood to be defined in an enlarged probability space with the use of the Skorohod representation.

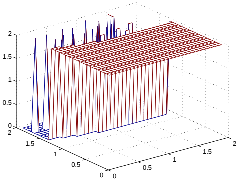
Numerical Approximations, Controlled Switching Diffusions, Games



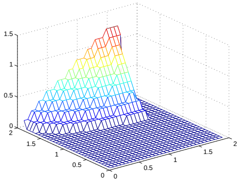
(a) $V^{h,+}(\cdot,\cdot,1)$



(b) $V^{h,+}(\cdot,\cdot,2)$



(c) $U_1(\cdot,\cdot,1)$: player1 1st



(d) $U_1(\cdot,\cdot,2)$ player1 1st

Numerics for Controlled Switching Diffusions

$$\begin{cases} X(t) = x + \int_0^t b(X(s), \alpha(s), u(s))ds + \int_0^t \sigma(X(s), \alpha(s))dw, \\ \alpha(t) \ \text{ continuous-time MC } \ \alpha(0) = i, \end{cases} \tag{34}$$

where $w(t)$ is a standard Brownian motion independent of the Markov chain $\alpha(t)$.

- Kushner & Dupuis, Springer, Markov chain approximation
- with Song & Zhang, Automatica (2006), regime-switching & jump diffusion

Controlled Switching Diffusions (cont.)

Given $B > 0$, define a stopping time as

$$\tau_B^{x,i,u} = \inf\{t : X^{x,i,u}(t) \notin (-B, B)\}.$$

Objective: choose control $u.$ to minimize the expected cost function

$$\begin{cases} J_i^B(x,u) & = \mathbf{E} \displaystyle\int_0^{\tau_B^{x,i,u}} f(X(s), \alpha(s), u(s))ds, \\ & \qquad \forall x \in (-B, B),\ i \in \mathscr{M}, \\ J_i^B(x,u) & = 0,\ \forall x \notin (-B, B),\ i \in \mathscr{M}, \end{cases} \tag{35}$$

where for each $i \in \mathscr{M}$, $f(\cdot, i, \cdot)$ is an appropriate function representing the running cost function.

For each $i \in \mathscr{M}$, the value function is given by

$$V^B(x,i) = \inf_{u \in \mathscr{U}} J^B(x,i,u), \tag{36}$$

where $\mathscr{U}$ is the space of all $\mathscr{F}_t$-adapted controls taking values on a compact set U.

Formally, the value functions satisfy Hamilton-Jacobi-Bellman (HJB) equations,

$$\begin{cases} \inf_{u \in U} \{L^u V^B(x,i) + f(x,i,u)\} = 0, & \forall x \in (-B,B), i \in \mathscr{M}, \\ V^B(x,i) = 0, & \forall x \notin (-B,B), i \in \mathscr{M}, \end{cases} \tag{37}$$

where

$$L^u \varphi(x,i) = \frac{1}{2}\sigma^2(x,i)\frac{d^2\varphi(x,i)}{dx^2} + b(x,i,u)\frac{d\varphi(x,i)}{dx} + \sum_{j \in \mathscr{M}} q_{ij}\varphi(x,j).$$

Algorithm

- $h > 0$: discretization parameter.
- $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$. Let $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathscr{M}$
- $p^h((x,i),(y,j)|u)$: transition probabilities from $(x,i) \in S_h \times \mathscr{M}$ to $(y,j) \in S_h \times \mathscr{M}$, for $u \in U$.

Then, $\bar{V}^{B,h}(x,i)$, the discretization of $V^B(x,i)$ with step size $h > 0$, is the solution of

$$\begin{cases} \inf_{u \in U} \{L_h^u \bar{V}^{B,h}(x,i) + f(x,i,u)\} = 0, & \forall x \in (-B,B)_h,\ i \in \mathscr{M}, \\ \bar{V}^{B,h}(x,i) = 0, & \forall x \notin (-B,B)_h,\ i \in \mathscr{M}, \end{cases} \tag{38}$$

$$(-B,B)_h = (-B,B) \cap S_h, \ \ [-B,B]_h = (-B,B)_h \cup \{B,-B\}. \tag{39}$$

Rates of Convergence

Theorem

Under suitable conditions, $\exists \gamma > 2$ and $\rho \in (0,1]$ s.t. the Markov chain approximation algorithm converges at the rate $(\gamma - 2) \wedge \rho \wedge \frac{1}{2}$. That is,

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \leq K h^{\frac{1}{2} \wedge \rho \wedge (\gamma - 2)}, \quad \forall (i,x) \in \mathscr{M} \times G.$$

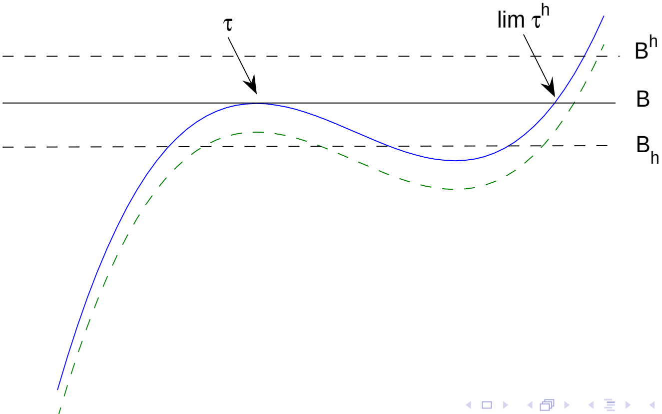
- Note that $\gamma \in (2,3]$ comes from Markov chain $\approx$, ρ is the Hölder exponent of the cost function.
- PDE approach for controlled diffusions (finite difference approx of PDEs)
  - Menaldi, SIAM J. Control Optim. (1989)
  - Krylov, Probab. Theory Related Fields, (2000)
  - Dong & N.V. Krylov, Appl. Math Optim.
- we use probabilistic approach for controlled switching diffusions
  - with Q.S. Song, SIAM J. Control Optim. (2009)

Main Ideas

- Use relaxed controls (measures)
- Construct strong approximation
- Consider boundary perturbations
  - usual notion of cost $J_i(x, \widetilde{m})$;
  - ours $J_i^{B}(x, \widetilde{m})$

Tangency Problem

- τ and τ^h: the first hitting time of $X(t)$ and $x^h(t)$ to the boundary.
- Objective: $\approx \mathbf{E}\tau$ by $\mathbf{E}\tau^h$
- In the Figure, $\tau^h \not\to \tau$, even though $x^h(\cdot)$ converges to $X(\cdot)$.
- Q: extra conditions needed?



Concluding Remarks

In this talk, we

- presented several switching diffusion examples
- considered recurrence, ergodicity, stability etc.
- presented noise suppression
- considered numerical algorithms for control and game problems

Further work:

- rates of convergence for games
- large deviations
- null-recurrent switching diffusion systems
- discrete-time counter part–Markov modulated random sequences
- ...

Thank you