One approach to solve a Monotone Nonlinear Boundary Problem

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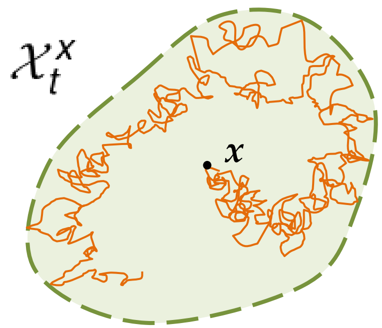
What is the stochastic control problem?

Let's concentrate our efforts towards avoiding these types of problems

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u(x) = 0, & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial \vec{n}}(x) + \lambda |u(x)|^{\mathrm{k}-1} u(x) = \Phi(x), & \text{on } \partial\Omega,\ \lambda > 0 \text{ and } \mathrm{k} > 1. \end{cases}$$

We focus our attention on discount rate (as control) of cost function when the stochastic process (SRBM) evolves on the boundary . . .



(SRBM)

Soriano, 02

$$u(x) = \mathbb{E}_x \left[\int_0^\infty \Phi\left(\mathcal{X}_t^x\right) \exp\left(-\int_0^t \lambda |u|^{\mathrm{k}-1} \left(\mathcal{X}_s^x\right) \mathrm{dL}_s^x \right) \mathrm{dL}_t^x \right]$$

Plan of this talk

Part I : Control Problem

Sketch of proof ...

DPP.

Part II : A fixed point problem

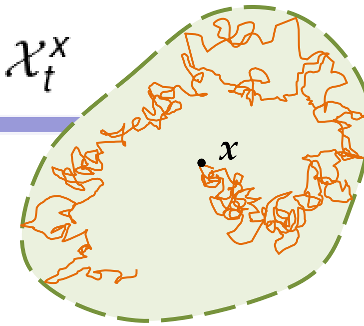
A Schauder Theorem application

Part III : Extensions.

More general problem.

Applications.

Some applications.

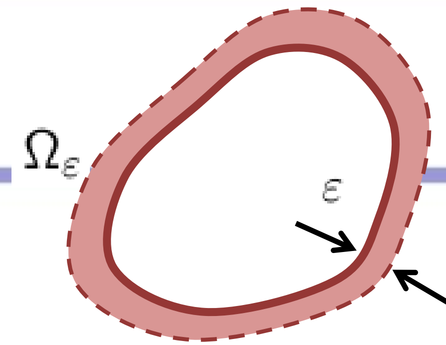


The stochastic process.

Scenario

- For each $x \in \overline{\Omega}$ initial state (deterministic) there is
  (SRBM) $\mathcal{X}_t^x = x + \sqrt{2}\mathcal{B}_t - \mathcal{K}_t^x, \quad 0 \leq t \leq \tau_x = \infty,$
  a stochastic process (random state).
- The exit time is $\tau_x = \inf\{t \geq 0 \quad ; \quad \mathcal{X}_t^x \notin \Omega\}$.
- Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be an open, bounded subset, for $x \in \Omega$ we can see that $\mathcal{X}_t^x \in \Omega, \forall t$.
- Moreover, if $\partial\Omega$ is smooth enough ($\partial\Omega \in \mathcal{C}^3$) then the bounded variation s.p. (T) $\mathcal{K}_t^x = \int_0^t \vec{n}(\mathcal{X}_s^x)\,\mathrm{dL}_s^x$ (Tanaka's formula [Ta]).
- Under these assumptions

$$(\mathrm{SRBM}) \qquad \mathcal{X}_t^x = x + \sqrt{2}\mathcal{B}_t - \int_0^t \vec{n}(\mathcal{X}_s^x)\,\mathrm{dL}_s^x,$$



Local time:definition and properties

The Local Time of (SRBM) on $\partial\Omega$ is defined as:

$$\mathrm{L}_t^x = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_0^t \vec{n}(\mathcal{X}_s^x)\, \mathrm{d}\mathrm{L}_s^x \text{ where } \Omega_\varepsilon = \left\{x \in \overline{\Omega} : \mathrm{d}(x, \partial\Omega) \leq \varepsilon\right\}.$$

Properties

- It is an additive functional of SRBM (with probability 1 it is increasing only when $\mathcal{X}_t^x \in \partial\Omega$).
- If $\mathbb{E}_x[\mathrm{L}_t] = \int_0^t \int_{\partial\Omega} p(s,x,y)\sigma(\mathrm{d}y)\mathrm{d}s$, then

$$\sup_{x \in \overline{\Omega}} \mathbb{E}_x[\mathrm{L}_t^n] \leq \mathrm{K}^n t^{\frac{n}{2}}, \qquad n \in \mathbb{N}.$$

- $\mathbb{P}^x(\mathrm{L}_t > 0; \forall t > 0) = \begin{cases} 0 & x \in \Omega, \\ 1 & x \in \partial\Omega. \end{cases}$

The cost function

For each $v \in \mathcal{C}(\overline{\Omega})$, there is a total cost function which consists only of the boundary payment

$$\mathcal{J}_{v(\cdot)}(x) = \mathbb{E}_x \left[\int_0^\infty \Phi(\mathcal{X}_t^x) \exp\left(- \int_0^t \lambda |v|^{k-1} (\mathcal{X}_s^x) \, \mathrm{d}L_s^x \right) \mathrm{d}L_t^x \right]$$

where $v > \eta > 0$ and Φ is the boundary cost.
We are interested in the criteria of optimization

$$u(x) = \inf_{v(\cdot)} \mathcal{J}_{v(\cdot)}(x).$$

u is the so called optimal cost function or value function.

Proposition: Cost function $\mathcal{J}_{v(\cdot)}(x)$ solves the boundary problem

$$(\mathcal{P}_v) \qquad \begin{cases} -\Delta \mathcal{J}_v = 0, & \text{in } \Omega. \\ -\frac{\partial \mathcal{J}_v}{\partial \vec{n}} + \lambda |v|^{k-1} \mathcal{J}_v = \Phi, & \text{on } \partial\Omega. \end{cases}$$

Vasilis Papanicolao, 98

P.L.Lions, 95, Menaldi, Sznitman,...

Argument of proof: Dynamic Programming Principle

Sketch of proof:

In order to use DPP arguments we can show that the operator

$$\mathcal{S}_t\phi(x) = \mathbb{E}_x\left[\int_0^t \Phi\left(\mathcal{X}_t^x\right)\exp\left(-\int_0^t \lambda|v|^{k-1}\left(\mathcal{X}_s^x\right)\mathrm{d}\mathrm{L}_s^x\right)\mathrm{d}\mathrm{L}_t^x\right]$$

is a semigroup related to the infinitesimal generator

$$\mathcal{L}^v\phi(\cdot) = -\Delta\phi(\cdot) + \left\{-\frac{\partial\phi(\cdot)}{\partial\vec{n}} + \lambda|v|^{k-1}\phi(\cdot)\right\}\mathrm{I}_{\partial\Omega}(x)$$

ITO

Applying Ito's formula to $\mathcal{J}_v\left(\mathcal{X}_t^x\right)$, we have

$$\mathbb{E}_x\left[\mathcal{J}_v(x)\right] = \mathbb{E}_x\left[\mathcal{J}_v\left(\mathcal{X}_t^x\right)\right] - \int_0^t \mathcal{L}^v\mathcal{J}_v\left(\mathcal{X}_s^x\right)\mathrm{d}s,$$

then by property of local time we can see two cases.
If $x \in \mathrm{int}(\Omega)$, then there is some neighborhood in $\mathrm{int}(\Omega)$ and therefore $\mathrm{L}_t = 0$ (with prob. 1) we can construct the quotient and tends $t \downarrow 0$ in order to obtain

$$\lim_{t\downarrow 0}\frac{\mathcal{S}_t\left(\mathcal{J}_v\right) - \mathcal{S}_0\left(\mathcal{J}_v\right)}{t} = \mathcal{L}^v\mathcal{J}_v = -\Delta\mathcal{J}_v$$

I T O

On the other hand, if $x \in \partial\Omega$, property ♦♦ allows to obtain

$$\begin{array}{rcl} \dfrac{\mathcal{S}_t(\mathcal{J}_v) - \mathcal{S}_0(\mathcal{J}_v)}{t} & = & \Delta \mathcal{J}_v(\cdot) + \left\{ -\dfrac{\partial \mathcal{J}_v(\cdot)}{\partial \vec{n}} + \lambda |v|^{k-1} \mathcal{J}_v(\cdot) \right\} \mathrm{I}_{\partial\Omega}(x) + \dfrac{o(t)}{t} \\ \downarrow & & \downarrow \\ \mathcal{L}^v \mathcal{J}_v & = & -\dfrac{\partial \mathcal{J}_v(\cdot)}{\partial \vec{n}} + \lambda |v|^{k-1} \mathcal{J}_v(\cdot) \end{array}$$

D P P

For $x \in \Omega$ and property ♦♦♦, therefore it follows from Bellman's principle that

$$\mathcal{S}_t(\mathcal{J}_v) - \mathcal{S}_0(\mathcal{J}_v) = \mathbb{E}_x\left[\mathcal{J}_v\left(\mathcal{X}_t^x\right)\right] - \mathcal{J}_v(x) = \mathbb{E}_x \int_0^t \left[\mathcal{L}^v \mathcal{J}_v\left(\mathcal{X}_s^x\right)\right] ds$$

where we divide all the expressions by t and also we let t tend to zero, obtaining thereby the equation

$$-\Delta \mathcal{J}_v(x) = \mathcal{L}^v \mathcal{J}_v(x) = \lim_{t \downarrow 0} \frac{\mathcal{S}_t(\mathcal{J}_v) - \mathcal{S}_0(\mathcal{J}_v)}{t} = 0.$$

D P P

For $x \in \partial\Omega$ then $x \in \cap_{\varepsilon > 0} \overline{\Omega_\varepsilon}$ and if $t \downarrow 0$ and $\mathrm{L}_t > 0$ with probability 1, therefore obtain

$$\begin{array}{rcl} \dfrac{\mathcal{S}_t(\mathcal{J}_v) - \mathcal{S}_0(\mathcal{J}_v)}{t} & = & \dfrac{1}{t}\mathbb{E}_x\left[\displaystyle\int_0^t \phi\left(\mathcal{X}_t^x\right) \exp\left(-\int_0^t \lambda |v|^{k-1}\left(\mathcal{X}_s^x\right) d\mathrm{L}_s^x\right) d\mathrm{L}_t^x\right] \\ \downarrow & & \downarrow \\ \mathcal{L}^v \mathcal{J}_v & = & -\Phi(x) \end{array}$$

Finally, cost functions solve the boundary problem $(\mathcal{P})_v$. □

The Banach space

The main goal is to extend the above analysis to study the boundary value problem $(\mathcal{P})$. Our last arguments are available to obtain an implicit representation based on some application $\mathcal{T}$

$$v \mapsto \mathcal{T}v = \mathcal{J}_v \qquad \text{unique solution of auxiliary } (\mathcal{P}_v).$$

Clearly, a solution of $(\mathcal{P})$ is a fixed point $\mathcal{T}u = u$ or equivalently solution have this representation.

- Considering the Banach space $\mathcal{A} = \mathcal{C}\left(\overline{\Omega}\right)$ with the supremum norm and the application $\mathcal{T}$.
- Data are $\Phi \in \mathcal{C}\left(\partial\Omega\right)$ and $f \equiv 0 \in \mathcal{C}^{0,\alpha}\left(\overline{\Omega}\right)$.
  Under these conditions $\mathcal{J}_v \in \mathcal{C}^{0,\alpha}\left(\overline{\Omega}\right) \cap \mathcal{C}\left(\overline{\Omega}\right) \subset \mathcal{C}\left(\overline{\Omega}\right)$ controls are $v \in \mathcal{A}$ such that

$$\mathcal{T} : \mathcal{A} \longrightarrow \mathcal{A}.$$

- **Upper bound of $\mathcal{J}_v$.**

$$\mathcal{J}_{v(\cdot)}(x) = \mathbb{E}_x\left[\int_0^\infty \Phi\left(\mathcal{X}_t^x\right)\exp\left(-\int_0^t \lambda|v|^{k-1}\left(\mathcal{X}_s^x\right)\mathrm{d}L_s^x\right)\mathrm{d}L_t^x\right]$$

$$\leq \|\Phi\|_{L^\infty}\mathbb{E}_x\left[\exp\left(-\int_0^\infty \lambda\eta^{k-1}\mathrm{d}L_s^x\right)\right]$$

$\lim_{t\uparrow\infty} L_t^x = +\infty$

$$\|\mathcal{J}_v\|_{L^\infty(\overline{\Omega})} \leq \|\Phi\|_{L^\infty(\partial\Omega)}\frac{1}{\lambda\eta^{k-1}} \doteq \mathcal{R}.$$

Remark: $\mathcal{R}$ does not depend on $v \geq \eta > 0$. In particular

$$\mathcal{T}\left(\overline{B_{\mathcal{R}}(0)}\right) \subset \overline{B_{\mathcal{R}}(0)}. \quad \square$$

•• $\mathcal{T}$ is a Lipschitz continuous application.

For $v, \widehat{v} \in \mathcal{C}\left(\overline{\Omega}\right)$ and $\mathcal{I}(|v|^{k-1}, s) \doteq \exp\left(-\int_0^s \lambda |v|^{k-1}\left(\mathcal{X}_s^x\right) dL_s^x\right)$ we compute a bound for difference $\mathcal{J}_v(x) - \mathcal{J}_{\widehat{v}}(x) \leq$

$\exp\left(-\int_0^\infty y ds\right) - 1 \leq \left|\int_0^\infty y ds\right|$

$\|\cdots\|_{L^\infty(\overline{\Omega})}$

$$\leq \|\Phi\|_{L^\infty(\partial\Omega)} \mathbb{E}_x \left[\int_0^\infty \left\{\mathcal{I}(|v|^{k-1}, s) - \mathcal{I}(|\widehat{v}|^{k-1}, s)\right\} dL_s^x\right]$$

$$\leq \|\Phi\|_{L^\infty(\partial\Omega)} \mathbb{E}_x \left[\int_0^\infty \mathcal{I}\left(|v|^{k-1}, s\right)\left(\mathcal{X}_s^x\right) dL_s^x \cdot \left\{\int_0^s \lambda \left[|v|^{k-1} - |\widehat{v}|^{k-1}\right]\left(\mathcal{X}_r^x\right) dL_r^x\right\} dL_s^x\right]$$

$\|\cdots\|_{L^\infty(\overline{\Omega})}$

$$\leq \lambda \|\Phi\|_{L^\infty(\partial\Omega)} \||v|^{k-1} - |\widehat{v}|^{k-1}\|_{L^\infty(\partial\Omega)} \cdot \mathbb{E}_x \left[\int_0^\infty \mathcal{I}\left(|v|^{k-1}, s\right) L_s^x dL_s^x\right]$$

$$= \lambda \|\Phi\|_{L^\infty(\partial\Omega)} \||v|^{k-1} - |\widehat{v}|^{k-1}\|_{L^\infty(\partial\Omega)} \cdot \frac{1}{\lambda \eta^{k-1}}$$

$\lim_{t\uparrow\infty} L_t^x = +\infty$, $v(\cdot) \geq \eta > 0$

($v \geq \eta > 0$ is independent of v and x).

$$\|\mathcal{J}_v - \mathcal{J}_{\widehat{v}}\|_{L^\infty(\partial\Omega)} \leq \lambda \|\Phi\|_{L^\infty(\partial\Omega)} \|v - \widehat{v}\|_{L^\infty(\partial\Omega)} \cdot \frac{1}{\lambda \eta^{k-1}} C(k, \mathcal{R}). \quad \square$$

Simons [Si] 90, Luc Tartar's inequality provides us arguments to conclude . . .

$1 < k \leq 2$, then $\mathcal{T}$ is Hölder continuous map.
$2 \leq k$, then $\mathcal{T}$ is Lipschitz continuous map.

A fixed point problem.

• • • Existence.
Schauder Theorem enables us to obtain a solution

$$u \in \overline{B_{\mathcal{R}}(0)} \subset \mathcal{A} \qquad \text{such that}$$

$$\mathcal{T}u = u. \quad \square$$

• • •• Uniqueness.
If do you have two solutions u_1 and u_2 by DP both are solutions of $(\mathcal{P}_v)$ and this problem have had uniqueness, then $u_1 \equiv u_2$ and can be written as our representation formula.

Part III Extensions.

a) The stochastic process $\mathcal{X}_t^x$ can be substituted by a more general process as the solution of

$$\text{(SDE)} \qquad \mathcal{X}_t^x = x + \int_0^t b\left(\mathcal{X}_s^x, \alpha\right) \mathrm{d}s + \int_0^t \sigma\left(\mathcal{X}_s^x, \alpha\right) \mathrm{d}\mathcal{B}s - \mathcal{K}_t^x, \qquad t \geq 0.$$

In this case,

$$\mathcal{J}_{v,\alpha}(x) = \mathbb{E}_x \Big[\int_0^{\tau_x} f\left(\mathcal{X}_t^x, \alpha\right) \exp\left(- \int_0^t \lambda |v|^{\mathrm{m}-1} \left(\mathcal{X}_s^x\right) \mathrm{d}_r \right) \mathrm{d}t + \\ + \int_0^\infty \Phi\left(\mathcal{X}_t^x, \alpha\right) \exp\left(- \int_0^t \lambda |v|^{\mathrm{k}-1} \left(\mathcal{X}_s^x\right) \mathrm{d}s \right) \exp\left(- \int_0^t \lambda |v|^{\mathrm{k}-1} \left(\mathcal{X}_s^x\right) \mathrm{d}\mathrm{L}_s^x \right) \mathrm{d}\mathrm{L}_t^x \Big]$$

solves the problem $(\mathcal{P})_{\mathbf{HJB}}$:

$$\begin{cases} \sup_{\alpha \in \mathcal{A}} \left\{ - \sum_{i;j=1}^n a_{ij}(x,\alpha) \dfrac{\partial^2 \mathcal{J}_v}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,\alpha) \dfrac{\partial \mathcal{J}_v}{\partial x_i} + - f(x,\alpha) \right\} + \widetilde{\lambda} |v(x)|^{\mathrm{m}-1} \mathcal{J}_v = 0, \\ \sup_{\alpha \in \mathcal{A}} \left\{ \sum_{i=1}^n \overline{\mu}_i(x,\alpha) \dfrac{\partial \mathcal{J}_v}{\partial x_i} - \Phi(x,\alpha) \right\} + \lambda |v(x)|^{\mathrm{k}-1} \mathcal{J}_v = 0, \qquad x \in \partial\Omega. \end{cases}$$

Now, the optimization criteria is

$$u(x) = \inf_{\alpha \in \mathcal{A}} \mathcal{J}_{v,\alpha}(x). \qquad \square$$

G.Diaz, R.Cabezas, ... (2008)

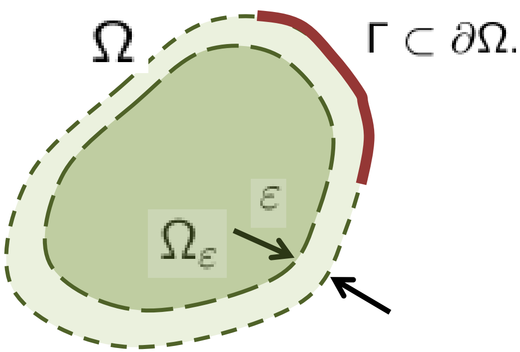
b) Blow up on compact subsets on the boundary $\Gamma \subset \partial\Omega$.
In order to analyze blow up on compact region $\Gamma \subset \partial\Omega$.

Internal approach.

An idea is to approach across internal subsets Ω_ε, $\varepsilon > 0$, where there are solutions u_ε. If there is uniform convergence in compact subsets of Ω, then it is to be hoped that

$$u_\varepsilon \longrightarrow u, \qquad \varepsilon \downarrow 0,$$

where u denotes a solution of problem $(\mathcal{P})_{\mathbf{HJB}}$



Boundary approach.

It consists in some approximations to the boundary condition by truncated solutions.

Some applications.

Radioactive-cooling problems. The steady state temperature distribution in various radiating bodies or gases lead to problems

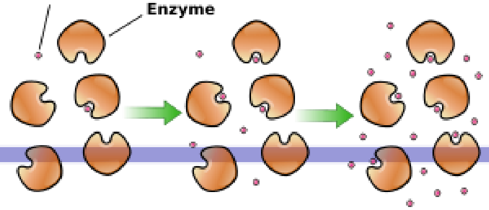
$$\text{(RC)} \begin{cases} \nabla\left(\kappa(x)\nabla \mathrm{T}\right) = \sigma(x)\mathrm{T}^4, & x \in D, \\ \kappa(x)\frac{\partial \mathrm{T}}{\partial \vec{n}} = \alpha(x,\mathrm{T})\left[\mathrm{T}_0(x) - \mathrm{T}\right], & x \in \partial D. \end{cases}$$

The thermal conductivity $\kappa(x) = \kappa$ constant (to fix ideas), Boltzmann factor $\sigma(x)$, and heat transfer coefficient $\alpha(x,\mathrm{T})$ are all positive as is imposed external temperature $\mathrm{T}_0(x)$. Then the solution is

$$\mathrm{T}(x) = \frac{1}{\kappa}\mathbb{E}_x\left[\int_0^\infty \alpha\left(\mathcal{X}_t^x,\mathrm{T}\right)\mathrm{T}_0\left(\mathcal{X}_t^x\right)\exp\left(-\frac{1}{\kappa}\int_0^t \alpha\left(\mathcal{X}_t^x,\mathrm{T}\right)\mathrm{dL}_s^x\right)\right.$$
$$\left.\exp\left(-\frac{1}{\kappa}\int_0^t \sigma\left(\mathcal{X}_t^x\right)\mathrm{T}^3\left(\mathcal{X}_t^x\right)\mathrm{d}s\right)\mathrm{dL}_t^x\right]$$

Stefan Law. $\kappa(x)\frac{\partial \mathrm{T}}{\partial \vec{n}} = \sigma(x,\mathrm{T})\left[\mathrm{T}_0^4(x) - \mathrm{T}^4\right],$ $x \in \partial D$, then the solution is

$$\mathrm{T}(x) = \frac{1}{\kappa}\mathbb{E}_x\left[\int_0^\infty \sigma\left(\mathcal{X}_t^x\right)\mathrm{T}^4\left(\mathcal{X}_t^x\right)\exp\left(-\frac{1}{\kappa}\int_0^t \sigma\left(\mathcal{X}_t^x,\mathrm{T}\right)\mathrm{dL}_s^x\right)\right.$$
$$\left.\exp\left(-\frac{1}{\kappa}\int_0^t \sigma\left(\mathcal{X}_t^x\right)\mathrm{T}^3\left(\mathcal{X}_t^x\right)\mathrm{d}s\right)\mathrm{dL}_t^x\right]. \quad \square$$



Some applications.

Diffusion kinetics of enzyme problems. The diffusion-kinetics eq. governing the steady-state concentration $\mathrm{C}(x)$ of some substrate in an enzyme-catalyzed reaction has the form
$\nabla \cdot (\mathrm{D}(x)\nabla \mathrm{C}(x)) = f(x, \mathrm{C})$.
$\mathrm{D}(x) > 0$ is the molecular diffusion coefficient of the substrate in a medium containing some continuous distribution of bacteria.
f is the rate of the enzyme-substrate reaction. If the domain S of interest (a cell) has surface ∂S consisting of a semipermeable membrane, then on this surface we have $\mathrm{D}\frac{\partial \mathrm{C}}{\partial \vec{n}} = h\,[\mathrm{C}_0 - \mathrm{C}]$,
$h(x) > 0$ is the permeability of the membrane,
$\mathrm{C}_0(x) > 0$ represents the external concentration of substrate, in this case we assume the reaction rate is given by the Michalis-Menten theory: $f(x, \mathrm{C}) = \dfrac{\varepsilon^{-1}\mathrm{C}}{\mathrm{C}+k}$,
$k > 0$ is the Michalis constant and $\varepsilon > 0$. Only the positive solutions are of physical interest. In this case the fixed point treatment is considered on some concave nonlinearity (which is a dual problem). Now, to fix $\mathrm{D}(x) = \mathrm{D}$ uniformly on the states x, the best candidate is precisely

$$\mathrm{C}(x) = \mathbb{E}_x\left[\frac{1}{\mathrm{D}}\int_0^{\infty} \mathrm{h}\left(\mathcal{X}_t^x\right)\mathrm{C}_0\left(\mathcal{X}_t^x\right)\exp\left(-\frac{1}{\mathrm{D}}\int_0^t \mathrm{h}\left(\mathcal{X}_s^x\right)\mathrm{dL}_s^x\right)\right.$$
$$\left.\exp\left(-\int_0^t \frac{\varepsilon^{-1}}{\mathrm{C}k}\left(\mathcal{X}_s^x\right)\mathrm{d}s\right)\mathrm{dL}_t^x\right]$$

In both examples is verified the estimation $0 \leq \mathrm{C}(x) \leq \mathrm{C}_0(x)$. □

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Argument of proof: Equality Boundary Conditions

Boundary condition.

ITO's rule enables us to obtain from $\mathcal{J}_v\left(\mathcal{X}_t^x\right)$ two equivalent expressions:

$$\mathcal{J}_v\left(\mathcal{X}_t^x\right)-\mathcal{J}_v\left(\mathcal{X}_0^x\right)=\int_0^t \nabla\mathcal{J}_v\left(\mathcal{X}_t^x\right)\mathrm{d}\mathcal{B}_s-\int_0^t \nabla\mathcal{J}_v\left(\mathcal{X}_s^x\right)\cdot\vec{n}\left(\mathcal{X}_s^x\right)\mathrm{dL}_s+\int_0^t \Delta\mathcal{J}_v\left(\mathcal{X}_s^x\right)\mathrm{d}s$$

$$\int_0^t \nabla\mathcal{J}_v\left(\mathcal{X}_t^x\right)\mathrm{d}\mathcal{B}_s=\mathcal{J}_v\left(\mathcal{X}_s^x\right)-\mathcal{J}_v\left(\mathcal{X}_0^x\right)-\int_0^t \Delta\mathcal{J}_v\left(\mathcal{X}_s^x\right)\mathrm{d}s+\int_0^t \nabla\mathcal{J}_v\left(\mathcal{X}_s^x\right)\cdot\vec{n}\left(\mathcal{X}_s^x\right)\mathrm{dL}_s$$

Right side is a continue martingale $\mathcal{M}^v(t)$ which should be compared with martingale related to our boundary data Ψ on $\partial\Omega$:

$$\mathcal{M}_\Psi^v(t)=\mathcal{J}_v\left(\mathcal{X}_t^x\right)-\mathcal{J}_v\left(\mathcal{X}_0^x\right)+\int_0^t \Psi\left(\mathcal{X}_s^x\right)\mathrm{dL}_s-\int_0^t \Delta\mathcal{J}_v\left(\mathcal{X}_s^x\right)\mathrm{d}s.$$

in order to obtain

$$\underbrace{\mathcal{M}^v(t)-\mathcal{M}_\Psi^v(t)}_{\mathbb{P}^x\texttt{-continuous martingale}} = \underbrace{\int_0^t\left(\frac{\partial\mathcal{J}_v}{\partial\vec{n}}-\Psi\right)\left(\mathcal{X}_s^x\right)\mathrm{dL}_s}_{\texttt{Bounded variation process}}$$

from uniqueness of Doob-Meyer descomposition $\int_0^t\left(\frac{\partial\mathcal{J}_v}{\partial\vec{n}}-\Psi\right)\left(\mathcal{X}_s^x\right)\mathrm{dL}_s=0\quad\mathbb{P}^x$-c.s.

If there is some $x_0 \in \partial\Omega$ such that $\frac{\partial \mathcal{J}_v}{\partial \vec{n}}(x_0) \neq \Psi(x_0)$, then continuity permits us concluded that there is V_δ and $\eta > 0$ such that

$$\frac{\partial \mathcal{J}_v}{\partial \vec{n}}(x) - \Psi(x) > \eta > 0 \qquad \forall x \in \mathrm{V}_\delta$$

Let $\tau_\delta = \inf\left\{\mathcal{X}_t^x \in \partial\Omega \setminus \mathrm{V}_\delta\right\}$ be an exit time on the rest of the boundary, it is positive from continuity of trajectories. We can choice $t_0 > 0$ such that $\mathbb{P}^*(\tau_\delta > t_0) > 0$, then

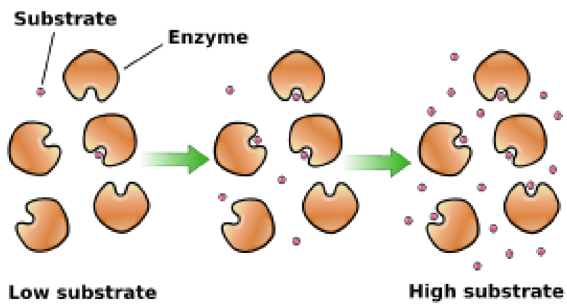
$$0 = \int_0^t \left|\frac{\partial \mathcal{J}_v}{\partial \vec{n}}(x) - \Psi\right| \left(\mathcal{X}_s^x\right) \mathrm{dL}_s \geq \eta \mathrm{L}(t_0) > 0$$

or $\mathrm{L}(t_0) = 0$ which contradicts ◆◆◆. Therefore,

$$\frac{\partial \mathcal{J}_v}{\partial \vec{n}}(x) - \Psi \qquad x \in \partial\Omega$$

◂ Return

As larger amounts of substrate are added to a reaction, the available enzyme binding sites become filled to the limit of V_{max}. Beyond this limit the enzyme is saturated with substrate and the reaction rate ceases to increase.



Saturation curve for an enzyme showing the relation between the concentration of substrate and rate