Nash equilibria in dynamic potential games via optimal control problems

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1. Introduction

Example 1.1 (The Stochastic Lake Game, Dechert and O'Donnell [2]). Each player $i = 1, \ldots, N$ solves the problem

$$\max_{\{u_{it}\}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t [v_i(u_{it}) - x_t^2] \tag{1.1}$$

subject to

$$x_{t+1} = h(x_t) + (u_{it} + U_{it}) Z_t, \tag{1.2}$$

where player i takes $U_{it} := \sum_{j \neq i} u_{jt}$ $(t = 0, 1, \ldots)$ as given.

Consider the following optimal control problem (OCP). Given the dynamics (1.2), maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t G(u_{1t}, \ldots, u_{Nt}, x_t) \tag{1.3}$$

where G is a **potential function**, given by

$$G(u_1, \dots, u_N, x) := \sum_{i=1}^{N} v_i(u_i) - x^2. \tag{1.4}$$

Proposition 1.2. *A solution to this OCP is a Nash equilibrium of the Stochastic Lake Game* (1.1)–(1.2).

The proof is based on Dechert [1].

Notation

Integrals are line integrals. That is, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable with component functions $f_1, f_2, \dots, f_n$ and $\phi : [0,1] \to \mathbb{R}^n$ is a C^1 function with components $\phi_1, \phi_2, \dots, \phi_n$, then

$$\int_{\phi(0)}^{\phi(1)} f(x)dx := \int_0^1 \left[\sum_{i=1}^{n} f_i(\phi(t)) \frac{d\phi_i}{dt} \right] dt.$$

The function f is said to be **exact** when this integral does not depend on the path ϕ. A necessary and sufficient condition for a C^1 function f to be exact is that

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \quad \text{for } i, j = 1, \ldots, n.$$

If $f : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a matrix of functions f_{ij} $(i, j = 1, \ldots, n)$, then $\int_{\phi(0)}^{\phi(1)} f(x)dx$ is a vector with components

$$\int_0^1 \left[\sum_{j=1}^n f_{ij}(\phi(t)) \frac{d\phi_j}{dt} \right] dt \quad \text{for } i = 1, \ldots, n.$$

Partial derivatives $\partial g / \partial x_i$ of a function $g : \mathbb{R}^n \to \mathbb{R}$ are also denoted by $\partial_{x_i} g$. The (row) gradient vector is written as $\partial_x g$. Finally, the transpose of a matrix A is denoted by A^*.

2. A related inverse problem

2.1. The control model

Consider the following performance index

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t G(x_t, x_{t+1}, \xi_t)\right], \tag{2.1}$$

where $\{x_t\}$ is a sequence in $\mathbb{R}^n$, $0 < \beta < 1$ is a discount factor, and $\{\xi_t\}$ is a sequence of i.i.d. random variables with values in a Borel set $S \subseteq \mathbb{R}^m$, and common distribution μ. The (deterministic) initial state x_0 and the initial value $\xi_0 = s_0$ are given. Each x_{t+1} will be chosen at time t after ξ_t has been observed. If ξ_t takes the value s_t and x_{t+1} belongs to the so–called **feasible set** $\Gamma(x_t, s_t)$, for all $t = 0, 1, 2, \ldots$, then the sequence $\{x_t\}$ is called a **feasible plan**.

Direct problem: Choose a feasible plan $\{x_t\}$ to maximize (2.1).

This kind of OCPs have been widely studied in economics; see, for instance, Stokey and Lucas [6, Chapter 9].

Assumption 2.1. (a) *The correspondence Γ is nonempty–valued. The graph of Γ is measurable, and Γ has a measurable selection.*

(b) *The function G in (2.1) is measurable and, moreover, there are measurable functions $L_t : \mathbb{R}^n \times S \to \mathbb{R}_+$ (with $\mathbb{R}_+ := [0, \infty)$) such that*

$$|G(x_t, x_{t+1}, \xi_t)| \leq L_t(x_0, s_0) \quad \forall t = 0, 1, 2, \ldots,$$

for each feasible plan $\{x_t\}$, and

$$\sum_{t=0}^{\infty} \beta^t L_t(x_0, s_0) < \infty.$$

Necessary conditions

Suppose that $\{\hat{x}_t\}$ is a solution to the OCP and $\hat{x}_{t+1}$ is an interior point of the set $\Gamma(\hat{x}_t, s_t)$ for all $t = 0, 1, \ldots$. Then $\{\hat{x}_t\}$ must satisfy the *stochastic Euler equation* (SEE)

$$\partial_{x_t} G(\hat{x}_{t-1}, \hat{x}_t, s_{t-1}) + \beta \mathbb{E}\left[\partial_{x_t} G(\hat{x}_t, \hat{x}_{t+1}, \xi_t)\right] = 0 \quad \forall t = 1, 2, \ldots, \tag{2.2}$$

whenever the derivative and the expectation operator can be interchanged. We refer to Stokey and Lucas [6, Chapter 9] for details.

Remark 2.2. In many OCPs it is considered a performance index of the form

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t g(x_t, u_t)\right]$$

where u_t is a control variable, and x_t is the state variable. The initial state x_0 is given, and the state evolves according to the law

$$x_{t+1} = h(x_t, u_t, \xi_t).$$

The model described above is less restrictive than it might seem at first sight, because in most cases, the latter formulation can be rewritten in the form (2.1); see, for instance, Sydsæter et al. [7, Section 12.2].

Example 2.3 (A cake–eating problem)**.** Consider a system in which the state variable x_t denotes the stock of a certain nonrenewable resource at time t. The initial state $x_0 > 0$ is given and the control variable c_t is the consumption at time t. Thus, the evolution of the system is given by

$$x_{t+1} = x_t - c_t, \quad t = 0, 1, \ldots.$$

Let U be a concave and increasing *utility function* defined on the control set

$[0, x_0]$. The OCP is to choose a sequence $\{c_t\}$ that maximizes the total discounted utility of consumption

$$\sum_{t=0}^{\infty} \beta^t U(c_t).$$

Equivalently, we wish to choose a sequence $\{x_t\}$ that maximizes

$$\sum_{t=0}^{\infty} \beta^t U(x_t - x_{t+1}). \tag{2.3}$$

Suppose that the utility function in (2.3) is a CES (constant elasticity of substitution) function $U(c) = (1-\sigma)^{-1} c^{1-\sigma}$, where $0 < \sigma < 1$. Then the Euler equation is

$$-\frac{1}{(x_{t-1} - x_t)^{\sigma}} + \beta \frac{1}{(x_t - x_{t+1})^{\sigma}} = 0, \quad x_0 \text{ given.} \tag{2.4}$$

2.2. The inverse optimal problem

Consider a sequence $\{\hat{x}_t\}$ that satisfies the difference equation

$$F(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \xi_{t-1}) = 0 \quad \forall t = 1, 2, \ldots, \tag{2.5}$$

where F is some continuously differentiable function (see the function F in Theorem 2.4), and the pair $(\hat{x}_0, \xi_0) = (x_0, s_0)$ is given.

Inverse problem: Find conditions under which there exists a function G such that $\{\hat{x}_t\}$ also satisfies (2.2).

This is an inverse problem because we want to find an objective function as in (2.1) when the SEE is given.

Theorem 2.4. *Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ be a continuously differentiable*

function. The following conditions (i) and (ii) are equivalent:

(i) *There exists a function* $G : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}$ *of class* $\mathcal{C}^2$ *such that*

$$F(x,y,z,\xi) = \partial_y G(x,y,\xi) + \beta \int \partial_y G(y,z,s)\, \mu(ds). \tag{2.6}$$

(ii) *There exist functions* $a, b : \mathbb{R}^n \times \mathbb{R}^n \times S \to \mathbb{R}^n$ *of class* $\mathcal{C}^1$ *such that*

(a) $F(x,y,z,\xi) = a(x,y,\xi) + \beta \int b(y,z,s)\, \mu(ds)$,

(b) $\partial_x a(x,y,\xi) = [\partial_y b(x,y,\xi)]^*$,

(c) $a(x,\cdot,\xi)$ *and* $b(\cdot,y,\xi)$ *are both exact,*

(d1) $\partial_x \int_{y_0}^{y} a(x,w,\xi)\, dw = \int_{y_0}^{y} [\partial_x a(x,w,\xi)]^*\, dw$,

(d2) $\partial_y \int_{x_0}^{x} b(w,y,\xi)\, dw = \int_{x_0}^{x} [\partial_y b(w,y,\xi)]^*\, dw$.

When conditions (a)–(d) of Theorem 2.4 hold, the function G is given by

$$G(x,y,\xi) := \int_{x_0}^{x} b(w,y,\xi)\,dw + \int_{y_0}^{y} a(x_0,w,\xi)\,dw. \tag{2.7}$$

See González–Sánchez and Hernández–Lerma [4, Theorem 3.1] for details.

Sufficient conditions

Theorem 2.6, below, gives conditions to ensure that a sequence satisfying the SEE (2.2) is indeed a maximizer of (2.1), where G is the function given by (2.7).

Dechert [1, Theorem 2] considers different conditions for the deterministic case. In particular, he supposes that

$$\sum_{t=0}^{\infty} \beta^t \|\partial_v G(\hat{x}_t, \hat{x}_{t+1})\| < \infty, \quad \text{for } v = x_t, x_{t+1}. \tag{2.8}$$

This assumption is not satisfied even for some elementary problems. For instance, we find in Example 2.7, below, that $\hat{x}_t = \beta^{t/\sigma} x_0$ ($t = 1, 2, \ldots$) solves the

OCP in Example 2.3, but

$$\sum_{t=0}^{\infty} \beta^t \|\partial_{x_t} G(\hat{x}_t, \hat{x}_{t+1})\| = \sum_{t=0}^{\infty} \beta^t \frac{1}{(\beta^{t/\sigma} x_0 - \beta^{(t+1)/\sigma} x_0)^\sigma}$$

is not finite; that is, (2.8) is not satisfied.

Assumption 2.5. (a) *The function G in (2.7) is concave in (x, y) and $\partial G / \partial x_i \geq 0$ for each $i = 1, 2, \ldots, n$.*

(b) *Each set $\Gamma(x, s)$ consists of nonnegative vectors only.*

Theorem 2.6. *Let F satisfy conditions* (a)–(d) *of Theorem* 2.4. *Consider the function G in* (2.7) *and a sequence $\{\hat{x}_t\}_{t=0}^{\infty}$ (in particular, $\hat{x}_0 = x$) that satisfies the difference equation*

$$F(\hat{x}_{t-1}, \hat{x}_t, \hat{x}_{t+1}, \xi_{t-1}) = 0, \quad t = 1, 2, \ldots.$$

Under Assumption 2.5 *and the* **transversality condition**

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\partial_{x_t} G(\hat{x}_t, \hat{x}_{t+1}, \xi_t)\right] \cdot \hat{x}_t = 0, \tag{2.9}$$

the sequence $\{\hat{x}_t\}_{t=0}^{\infty}$ *is a solution to the problem*

$$\max \left\{ \mathbb{E} \sum_{t=0}^{\infty} \beta^t G(x_t, x_{t+1}, \xi_t) \,\middle|\, x_0 = x,\ \xi_0 = s_0 \text{ given} \right\}.$$

See Kamihigashi [5] for an explanation of the transversality condition (2.9) and the relationship to some applications in economics.

Example 2.7. The Euler equation for the Cake–eating problem in Example 2.3

$$x_{t+1} - (1 + \beta^{1/\sigma})x_t + \beta^{1/\sigma} x_{t-1} = 0, \quad x_0 \text{ given},$$

is a second order difference equation. Its solution is $\hat{x}_t = c_1 + c_2 \beta^{t/\sigma}$ $(t = 0, 1, 2, \ldots)$, for some constants c_1, c_2. To determine these two constants, we have to use both the initial condition and the transversality condition (2.9). We obtain $c_1 = 0$ and $c_2 = x_0$. Therefore, the optimal Markov policy for consumption is

$$\hat{c}_t = (1 - \beta^{1/\sigma})\hat{x}_t, \quad t = 0, 1, 2, \ldots.$$

3. Dynamic games with a potential function

Consider a game with player i's reward function, $i = 1, 2, \ldots, N$, given by

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t r^i(x_t, x_{t+1}, \xi_t). \tag{3.1}$$

Each player i has to choose a sequence $\{x_t^i\}$ to maximize (3.1) given $\{x_t^j\}$ for $j \neq i$. The corresponding SEEs for an open–loop Nash equilibrium $\{\hat{x}_t\}$ are

$$\partial_{x_{it}} r^i(\hat{x}_{t-1}, \hat{x}_t, s_{t-1}) + \beta \mathbb{E}\left[\partial_{x_{it}} r^i(\hat{x}_t, \hat{x}_{t+1}, \xi_t)\right] = 0 \tag{3.2}$$

for each $i = 1, \ldots, N$, and for all $t = 0, 1, \ldots$.

Dynamic potential games

To find Nash equilibria, we can use the inverse problem described in Section 2.2. Replace the difference equation (2.5) by (3.2) and suppose that hypotheses of Theorems 2.4 and 2.6 hold. Then the previous game is a **potential game**. That is, there is a **potential function** G such that a solution to the OCP

$$\max \left\{ \mathbb{E} \sum_{t=0}^{\infty} \beta^t G(x_t, x_{t+1}, \xi_t) \,\middle|\, x_0 = x,\ \xi_0 = s_0 \text{ given} \right\},$$

is also a Nash equilibrium of the game. The potential function is given by (2.7).

Example 3.1. Dockner et al. [3] consider a capital accumulation game with two players. The reward functions are

$$\begin{aligned} J^1(\cdot) &= \sum_{t=0}^{\infty} \beta^t [\pi^1(x_t, y_t) - I_t^1], \\ J^2(\cdot) &= \sum_{t=0}^{\infty} \beta^t [\pi^2(y_t, x_t) - I_t^2], \end{aligned}$$

and the dynamics

$$\begin{aligned} x_{t+1} &= I_t^1 + (1 - \delta_1) x_t, \\ y_{t+1} &= I_t^2 + (1 - \delta_2) y_t. \end{aligned}$$

The rewards for this game can be written in the form (3.1) as follows

$$\sum_{t=0}^{\infty} \beta^t [\pi^1(x_t, y_t) - x_{t+1} + (1-\delta_1)x_t],$$

$$\sum_{t=0}^{\infty} \beta^t [\pi^2(y_t, x_t) - y_{t+1} + (1-\delta_2)y_t].$$

This is a potential game if

$$\frac{\partial^2 \pi^1}{\partial x \partial y}(x, y) = \frac{\partial^2 \pi^2}{\partial x \partial y}(y, x). \tag{3.3}$$

When (3.3) holds, there exists a function π such that

$$\frac{\partial \pi}{\partial x}(x, y) = \frac{\partial \pi^1}{\partial x}(x, y), \quad \frac{\partial \pi}{\partial y}(x, y) = \frac{\partial \pi^2}{\partial y}(y, x).$$

A potential function G is given by

$$\pi(x_t, y_t) + (1-\delta_1)x_t + (1-\delta_2)y_t - x_{t+1} - y_{t+1}.$$

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