COMPARISON BETWEEN MARTINGALE METHODS AND DYNAMIC PROGRAMMING

Alain Bensoussan[1] Abel Cadenillas[2] Hyeng Keun Koo [3]
Jaeyoung Sung [4]

[1]International Center for Decision and Risk Analysis
University of Texas at Dallas
The Hong Kong Polytechnic University and Ajou University

[2]University of Alberta and Ajou University

[3]Ajou University

[4]Ajou University

OUTLINE I

OUTLINE II

5. STOCHASTIC CONTROL PROBLEM
   - THE CHOICE OF λ
   - OPTIMAL $\hat{v}(s)$
   - COMPARISON WITH DYNAMIC PROGRAMMING

6. FURTHER COMPARISON
   - DERIVATIVES OF THE VALUE FUNCTION
   - CONCLUDING REMARKS

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DYNAMIC PROGRAMMING

- Plays an overwhelming role in solving Stochastic Control Problems
- It is a theory of *sufficient* conditions of optimality
- Provides an optimal control as a feedback
- In non-smooth cases provides an equation for the value function (viscosity solution)

MARTINGALE METHODS

- Mathematical Finance considers specific problems of stochastic control
- Major example, Consumption-investment optimization problem, Karatzas, Shreve (1998)
- Can handle non-markovian cases
- Direct approach possible, thanks to the specific form of the problem
- In principle, equivalent to Dynamic Programming in the Markovian case

ECONOMIC FRAMEWORK

- Consider an investor or an entrepreneur, who increases his/her wealth thanks to personal efforts and adequate choice of projects
- Personal efforts have a deterministic positive impact on the wealth, but entail also a cost in the objective function of the investor
- Projects are scalable, with increasing expected return but also with increasing risk
- We consider a finite horizon problem

EVOLUTION OF WEALTH

Consider a probability space $(\Omega, \mathscr{A}, P)$ and a one-dimensional Wiener process $w(t)$. We denote by $\mathscr{F}^t$ the filtration generated by the Wiener process. The wealth $X(s)$ satisfies

$$dX = \delta u(s)ds + v(s)X(s)(\alpha ds + dw(s)), \quad s > t \qquad (1)$$
$$X(t) = x$$

We denote by $X_{xt}(s)$ the solution, to indicate initial conditions. The wealth depends on two controls, $u(s)$ the level of effort and $v(s)$ the scale of projects. They are adapted processes such that

$$u(s) > 0,\ E\int_t^T u(s)^2 ds < \infty,\ \ v(s)X(s) > 0,\ E\int_t^T (v(s)X(s))^2 ds < \infty$$

The constants $\delta$and α are strictly positive.

PAYOFF FUNCTIONAL I

We see that the effort has a direct deterministic positive impact on the wealth evolution. An increase of the scale of projects has a positive impact on the expected return, but elicits also an increase of the risk. The objective functional is defined as follows

$$J_{x,t}(u(.),v(.)) = E[F(X_{x,t}(T)) - \frac{1}{2}\int_t^T u(t)^2 dt] \tag{2}$$

The second term represents the desutility of the effort. The first term represents the utility of the final wealth. We assume

$$F(x) = \log(w + (x-K)^+) \tag{3}$$

PAYOFF FUNCTIONAL II

The agent is risk-averse. The number w (not to be confused with the Wiener process) is ≥ 0 . It represents a minimal subsistance level. The number K is a debt that the agent must reimburse at time T. We may associate this debt to an amount which is borrowed at time t. If the entrepreneur borrows L at the origin of the activity, then we have

$$x = x_0 + L$$

in which $x_0$represents the initial capital.

CREDIT RISK

The bank must determine the amount L in terms of K. This problem can also be treated in the present framework. It contributes to the theory of *credit risk*. We may add an additional degree of difficulty assuming that we do not know exactly α,but we know only that it is in a range $[\alpha_{0,}\alpha_1]$, with $\alpha_0 > 0$ and $\alpha_1 < \infty$.

BELLMAN EQUATION I

We define the value function

$$\Phi(x,t) = \sup_{u(.),v(.)} J_{x,t}(u(.),v(.)) \tag{4}$$

We expect the function Φ to be increasing and concave in x. The Bellman equation is easily written as

$$\frac{\partial \Phi}{\partial t} + \sup_u\left(-\frac{1}{2}u^2 + \delta u \frac{\partial \Phi}{\partial x}\right) + \sup_v\left(\alpha v x \frac{\partial \Phi}{\partial x} + \frac{1}{2}x^2 v^2 \frac{\partial^2 \Phi}{\partial x^2}\right) = 0 \tag{5}$$

with initial condition

$$\Phi(x,T) = F(x) \tag{6}$$

Rewriting Bellman equation I

We write Bellman equation as follows

$$\frac{\partial \Phi}{\partial t} + \frac{\delta^2}{2}(\frac{\partial \Phi}{\partial x})^2 - \frac{1}{2}\alpha^2 \frac{(\frac{\partial \Phi}{\partial x})^2}{\frac{\partial^2 \Phi}{\partial x^2}} = 0 \qquad (7)$$

after performimg the optimization in u and v.

Feedbacks

We then have the feedbacks

$$u(x,t) = \delta \frac{\partial \Phi}{\partial x}, \qquad xv(x,t) = -\alpha \frac{\frac{\partial \Phi}{\partial x}}{\frac{\partial^2 \Phi}{\partial x^2}} \tag{8}$$

These feedbacks satisfy the positivity conditions, if the solution is monotone increasing and concave.

Generic Solution I

Remarkably, in spite of its apparent complexity, Bellman equation has a generic solution, given by the following procedure

$$\Phi(x,t) = \Psi(\lambda(x,t),t) \tag{9}$$

where

$$\Psi(\lambda,t) = \frac{2}{\alpha^2}(A\log\lambda + (B\exp -\alpha^2 t + \frac{1}{2}\delta^2)\frac{\lambda^2}{2}) + At + L \tag{10}$$

and λ is defined by the equation

$$-\frac{A}{\lambda} + (B\exp -\alpha^2 t + \frac{1}{2}\delta^2)\lambda = \frac{\alpha^2}{2}x + C \tag{11}$$

in which A, B, C, L are arbitrary constants.

Conditions for solution I

The proof can be shown by direct checking. In particular, one checks that

$$\frac{\partial \Phi}{\partial x} = \lambda, \quad \frac{\partial \Phi}{\partial t} = A + \lambda^2 B \exp -\alpha^2 t \tag{12}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\alpha^2}{2} \frac{\lambda^2}{A + \lambda^2 (B \exp -\alpha^2 t + \frac{\delta^2}{2})} \tag{13}$$

CONDITIONS

We must take the positive solution of (11), which implies that A and $B \exp -\alpha^2 t + \frac{1}{2}\delta^2$ have the same sign. From the concavity condition we must have

$$A < 0, \quad -B \exp -\alpha^2 T > \frac{\delta^2}{2} \tag{14}$$

MATCHING THE FINAL CONDITION I

Unfortunately, a problem arises when we try to match the final condition

$$\Phi(x,T) = F(x) = \log(w + (x-K)^+)$$

We must have

$$\lambda(x,T) = \frac{\mathbb{1}_{x>K}}{w+x-K}$$

This expression is discontinuous at $x=K$. A matching is possible for $x>K$. We obtain

$$A = -\frac{\alpha^2}{2}, \quad B = -\frac{\delta^2}{2}\exp\alpha^2 T, \quad C = -\frac{\alpha^2}{2}(K-w)$$

MATCHING THE FINAL CONDITION II

However this matching will not be valid when $x < K$, where $\lambda(x, T) = 0$. If we match $\Phi(x, T)$ at $x > K$, we obtain $L = \frac{\alpha^2}{2} T$. However, the full matching is impossible. Therefore, the generic solution (9),(10),(11) of Bellman equation does not provide the value function.

COMMENT I

We cannot think either to a non-smooth solution of Bellman equation. Indeed, as we shall see the value function is smooth, for $t < T$. In spite of the smoothness, it will not satisfy Bellman equation.
IS THERE A CONTRADICTION?
No, because Dynamic Programming is a sufficient condition, not a necessary condition.

EXPRESSION

Note that (11) yields

$$\frac{1}{\lambda} = \frac{1}{2}(x - K + w + \sqrt{(x - K + w)^2 + 4\delta^2 \tilde{T}_t}) \quad (15)$$

where we have set

$$\tilde{T}_t = \frac{\exp \alpha^2(T - t) - 1}{\alpha^2}$$

CASE $w = 0$ I

Curiously, things improve when $w = 0$. A priori, things look worse, because

$$F(x) = \log(x - K), \text{ if } x > K; \qquad F(x) = -\infty, \text{ if } x > K$$

We must interpret $\lambda(x, T)$ as follows

$$\lambda(x, T) = \left| \begin{array}{ll} \dfrac{1}{x - K} & \text{if } x > K \\ +\infty & \text{if } x < K \end{array} \right.$$

EXPRESSION

We can check that

$$\frac{1}{\lambda} = \frac{1}{2}(x - K + \sqrt{(x-K)^2 + 4\delta^2 \tilde{T}_t})$$

When $t = T$ we get $\frac{1}{\lambda} = 0$, when $x < K$ and the matching is valid. In this case the generic solution is the value function.

NEW CONTROL PROBLEM I

We introduce the martingale $Z(s)$ defined by

$$\begin{aligned} dZ &= -\alpha Z dw(s), \; s > t \\ Z(t) &= 1 \end{aligned} \tag{16}$$

then, combining with the wealth equation, we get

$$\begin{aligned} d\,XZ &= \delta Z(s)u(s)ds + Z(s)X(s)(v(s)-\alpha)dw(s) \\ XZ(t) &= x \end{aligned} \tag{17}$$

FORMULATION

Hence, by integration

$$EX(T)Z(T) = x + \delta E \int_t^T Z(s)u(s)ds \tag{18}$$

The idea is to consider a control problem in which $X(T)$ and $u(.)$ are the decision variables. The objective is to optimize the payoff

$$\mathscr{J}(X(T), u(.)) = E[F(X(T)) - \frac{1}{2}\int_t^T u(s)^2 ds] \tag{19}$$

subject to the single constraint (18), which is called the budget equation.

COMMENTS

In the new problem, the control $v(.)$ has been discarded, as well as the evolution equation of the wealth (1) or equivalently (17). This is possible, because $v(.)$ does not appear explicitly in the objective functional (19), nor in the constraint (18).

Of course the two problems are not equivalent. In fact, we can claim that

$$\sup_{\text{budget equation}} \mathscr{J}(X(T), u(.)) \geq \Phi(x,t)$$

since the initial problem is equivalent to the new problem, reinstating the evolution (17) as a constraint. The new problem is no more a control problem, since there is no state evolution. $X(T)$ is any $\mathscr{F}^T$ random variable, and the budget equation is a constraint on this random variable.

EQUIVALENCE CONDITIONS I

Suppose we can solve the new problem and get a solution denoted by $\hat{X}(T)$, $\hat{u}(.)$ (note that it depends on the pair x, t). In the case it is possible to find $\hat{v}(.)$ such that $\hat{X}(T)$ is the final value of an evolution equation

$$d\hat{X} = \delta\hat{u}(s)ds + \hat{v}(s)\hat{X}(s)(\alpha ds + dw(s)), \quad s > t \tag{20}$$
$$\hat{X}(t) = x$$

then it means that the additional evolution equation, considered as a constraint is automatically satisfied. If this happens to be true, then the two problems are equivalent.

COMMENT

We note that, solving the new problem is a priori easier, since we have a single scalar constraint in a stochastic optimization problem, and not a stochastic control problem.

LAGRANGE MULTIPLIER I

We treat the budget equation with a Lagrange multiplier. We will denote it by λ, in fact $\lambda(x,t)$. Although we use the notation introduced in the case of Bellman equation to represent $\frac{\partial \Phi}{\partial x}$, an important observation will be that it will not coincide with the derivative in x of the value function, which is consistent with the statement that the generic solution of the Bellman equation does not coincide with the value function.

OPTIMIZATION

We then consider the problem, without constraint,

$$\sup_{X(T),u(.)} E[F(X(T)) - \frac{1}{2}\int_t^T u(s)^2 ds - \lambda X(T)Z(T) + \lambda\delta\int_t^T u(s)Z(s)ds] \tag{21}$$

In (21) $\lambda$is a constant, which we postulate will be positive, and there are no constraints on the pair $X(T)$, $u(.)$.

SOLUTION I

The optimization in $u(.)$ is straightforward. We get

$$\hat{u}(s) = \delta\lambda Z(s) \tag{22}$$

To obtain the optimal $X(T)$ we have to mazimize in y the function $F(y) - zy$ in y, where z is the fixed value $\lambda Z(T)$, which is positive. A quick analysis yields that the optimum of $F(y) - zy$ is attained at

$$\hat{s}(z) = (K - w + \frac{1}{z})\mathbb{1}_{z<\frac{1}{w}} \tag{23}$$

We observe that this function is discontinuous, at point $\frac{1}{w}$. If $w = 0$, it becomes continuous, equal to $K + \frac{1}{z}$.

Equation for λ I

We write the budget equation

$$x = E\hat{s}(\lambda Z(T))Z(T) - \lambda\delta^2 E\int_t^T Z^2(s)ds$$

A quick observation is that

$$E\int_t^T Z^2(s)ds = \tilde{T}_t = \frac{\exp\alpha^2(T-t)-1}{\alpha^2}$$

The term in $Z(T)$ is a little bit more involved. Eventually, we get the equation

$$x = G(\lambda, t) \tag{24}$$

EQUATION FOR λ II

with the definition

$$G(\lambda,t)=(K-w)N(-d_1(\lambda,t))+\frac{1}{\lambda}N(-d_2(\lambda,t))-\lambda\delta^2\tilde{T}_t \quad (25)$$

in which

$$d_1(\lambda,t)=\frac{1}{\alpha\sqrt{T-t}}\log\lambda w+\frac{\alpha\sqrt{T-t}}{2} \quad (26)$$

$$d_2(\lambda,t)=d_1(\lambda,t)-\alpha\sqrt{T-t} \quad (27)$$

and $N(x)$ represents the cumulative distribution function of the standard gaussian random variable.

PROPERTIES OF $G(\lambda, t)$ I

We have for $t < T$,we have

$$G(0,t) = +\infty,\; G(+\infty,t) = -\infty$$
$$\frac{\partial G}{\partial \lambda}(\lambda,t) < 0$$

The first two properties are immediate. The monotonicity is more elaborate. We write

$$G(\lambda,t) = KN(-d_1(\lambda,t)) + \frac{1}{\lambda}U(\log \lambda w, t) - \lambda\delta^2\tilde{T}_t$$

with

PROPERTIES OF $G(\lambda, t)$ II

$$U(x,t) = N\left(-\frac{1}{\alpha\sqrt{T-t}}x + \frac{\alpha\sqrt{T-t}}{2}\right) - $$

$$- \exp x\, N\left(-\frac{1}{\alpha\sqrt{T-t}}x - \frac{\alpha\sqrt{T-t}}{2}\right)$$

PROOF I

In the expression of $U(x,t)$, x is a real number. We note that

$$U(-\infty,t)=1,\; U(+\infty,t)=0$$

Moreover

$$\frac{\partial U(x,t)}{\partial x}=-\exp\, x\, N\left(-\frac{1}{\alpha\sqrt{T-t}}x-\frac{\alpha\sqrt{T-t}}{2}\right)<0$$

which implies also $U(x,t)>0$.

Proof

But then

$$\frac{\partial[\frac{1}{\lambda}U(\log\lambda w,t)]}{\partial\lambda}<0$$

The other terms in the expression of $G(\lambda,t)$ are also decreasing, which implies $G(\lambda,t)$ is decreasing.

DEFINITION OF $\lambda(x,t)$ I

It follows that the Lagrange multiplier $\lambda = \lambda(x,t)$ is uniquely defined by equation (24) and positive for any x and $t < T$. For $t = T$ we have

$$G(\lambda, T) = \left| \begin{array}{ll} 0 & \text{if } \lambda > \frac{1}{w} \\ K - w + \frac{1}{\lambda} & \text{if } \lambda < \frac{1}{w} \end{array} \right. \quad (28)$$

We observe again a discontinuity. Moreover, we can solve the equation $x = G(\lambda, T)$ only for $x > K$. Since one cannot get a finite Lagrange multiplier at time $t = T$,when $x < K$ we shall set

$$\lambda(x, T) = \left| \begin{array}{ll} \frac{1}{w + x - K} & \text{if } x > K \\ +\infty & \text{if } x \leq K \end{array} \right. \quad (29)$$

DEFINITION OF $\lambda(x,t)$ II

We then have

$$\hat{X}(T) = (K - w + \frac{1}{\lambda Z(T)})\mathbb{1}_{\lambda Z(T) < \frac{1}{w}} \tag{30}$$

OPTIMAL STATE TRAJECTORY I

In order to check that the new problem provides a solution of the initial control problem we must define a process $\hat{v}(s)$such that the trajectory defined by equation (20) has a final value $\hat{X}(T)$ which coincides with (30). It is useful to perform the following standard transformation

$$\tilde{w}(s) = w(s) + \alpha s, \qquad \mathscr{F}_t^s = \sigma(w(u) - w(t) | t \le u \le s)$$

CHANGE OF PROBABILITY

$$\frac{d\tilde{P}}{dP}\Big|_{\mathscr{F}_t^s} = Z(s)$$

then $\tilde{w}(s) - \tilde{w}(t)$ is a $\Omega, \mathscr{A}, \tilde{P}, \mathscr{F}_t^s$ standard Wiener process, and the evolution equation (20) becomes

$$d\hat{X} = \lambda\delta^2 Z(s)ds + \hat{v}(s)\hat{X}(s)d\tilde{w} \tag{31}$$

Optimal $\hat{X}(s)$ I

From (31) we obtain

$$\hat{X}(s) = \tilde{E}[\hat{X}(T) - \lambda\delta^2 \int_s^T Z(u)du|\mathscr{F}_t^s]$$

Since

$$dZ = -\alpha Z(s)d\tilde{w(s)} + \alpha^2 Z(s)ds$$

CALCULATION

we have

$$\tilde{E}[Z(u)|\mathscr{F}_t^s] = Z(s)\exp\alpha^2(u-s),\ \forall u \geq s \geq t$$

and

$$\tilde{E}[\int_s^T Z(u)du|\mathscr{F}_t^s] = Z(s)\tilde{T}_s$$

FORMULA I

We get

$$\hat{X}(s) = \tilde{E}[\hat{X}(T)|\mathscr{F}_t^s] - \lambda\delta^2 Z(s)\tilde{T}_s \tag{32}$$

From equation (30) we can compute the conditional expectation. We get the following important result

$$\hat{X}(s) = G(\lambda Z(s), s) \tag{33}$$

and for $s = t$ we recover formula (24).

P.D.E. I

A tedious calculation shows that $G(x,t)$is the solution of the P.D.E.

$$\frac{\partial G}{\partial t}+\alpha^2\left(x\frac{\partial G}{\partial x}+\frac{1}{2}x^2\frac{\partial^2 G}{\partial x^2}\right)=x\delta^2 \tag{34}$$

$$G(x,T)=\left|\begin{array}{ll} K-w+\frac{1}{x} & \text{if } xw<1 \\ 0 & \text{if } xw>1 \end{array}\right. \tag{35}$$

OPTIMAL FEEDBACKS I

We can use Ito's calculus to expand (33) and compare with equation (31). We can identify $\hat{v}(s)$ by

$$\hat{v}(s)\hat{X}(s) = -\alpha\lambda Z(s)\frac{\partial G}{\partial x}(\lambda Z(s), s) \tag{36}$$

Applying with $s = t$ we obtain the optimal feedbacks

$$\hat{u}(x,t) = \delta G^{-1}(x,t) = \delta\lambda(x,t) \tag{37}$$

$$\hat{v}(x,t) = -\frac{\alpha}{x}G^{-1}(x,t)\frac{\partial G}{\partial x}(G^{-1}(x,t),t) \tag{38}$$

$$= -\frac{\alpha}{x}\lambda(x,t)\frac{\partial G}{\partial x}(\lambda(x,t),t)$$

OPTIMAL FEEDBACKS II

Moreover the feedbacks satisfy the positivity conditions.
Since we have obtained $\hat{u}(s)$and $\hat{v}(s)$ we can claim that the new problem solution is also the solution of the initial stochastic control problem.

VALUE FUNCTION I

We can next compute the value function

$$\Phi(x,t) = E[F(\hat{X}(T)) - \frac{1}{2}\lambda^2\delta^2\tilde{T}_t] \tag{39}$$

In a way similar to that developed for the D.P. approach (see (9), we can write the value function as follows

$$\Phi(x,t) = \Psi(\lambda(x,t),t) \tag{40}$$

with

VALUE FUNCTION II

$$\Psi(\lambda,t) = \log w\, N(d_2(\lambda,t)) + (\frac{\alpha^2}{2}(T-t) - \log\lambda)N(-d_2(\lambda,t)) \tag{41}$$

$$+\frac{\alpha\sqrt{T-t}}{\sqrt{2\pi}}\exp -\frac{1}{2}(d_2(\lambda,t))^2 - \frac{\lambda^2\delta^2}{2}\tilde{T}_t$$

We note that

$$\Psi(\lambda,T) = \left| \begin{array}{ll} \log w & \text{if } \lambda w > 1 \\ \log\dfrac{1}{\lambda} & \text{if } \lambda w < 1 \end{array} \right.$$

and in view of the definition of $\lambda(x,T)$, see (29) we recover $\Phi(x,T) = F(x)$.

LAGRANGE MULTIPLIER I

Considering the formulas giving $\hat{u}(x,t)$ from Dynamic Programming and Martingale approach, they agree only whenever

$$\lambda(x,t) = \frac{\partial \Phi(x,t)}{\partial x}$$

We check that this is not true. Indeed,

$$\frac{\partial \Phi}{\partial x} = \frac{\dfrac{\partial \Psi}{\partial \lambda}}{\dfrac{\partial G}{\partial \lambda}}$$

COMPUTATION I

and

$$\frac{\partial \Psi}{\partial \lambda} = -\frac{1}{\lambda} - \lambda \delta^2 \tilde{T}_t + \frac{1}{\lambda} N(d_2)$$

$$\frac{\partial G}{\partial \lambda} = -\frac{1}{\lambda^2} - \delta^2 \tilde{T}_t + \frac{1}{\lambda^2} N(d_2) -$$

$$\frac{1}{\alpha \lambda \sqrt{2\pi(T-t)}} [(K-w)\exp -\frac{1}{2} d_1^2 + \frac{1}{\lambda} \exp -\frac{1}{2} d_2^2]$$

$$-\frac{1}{\alpha \lambda \sqrt{2\pi(T-t)}} [(K-w)\exp -\frac{1}{2} d_1^2 + \frac{1}{\lambda} \exp -\frac{1}{2} d_2^2]$$

and the ratio is not λ.

OBTAINING FIRST DERIVATIVES I

We have already obtained $\frac{\partial \Phi}{\partial x}$. We then notice that

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Psi}{\partial t} - \frac{\partial \Phi}{\partial x}\frac{\partial G}{\partial t}$$

and

$$\frac{\partial \Psi}{\partial t} = -\frac{\alpha^2}{2} + \lambda^2 \frac{\delta^2}{2}(1+\alpha^2 \tilde{T}_t) + \frac{\alpha^2}{2} N(d_2) - \frac{\alpha}{2\sqrt{2\pi(T-t)}} \exp -\frac{1}{2} d_2^2$$

DERIVATIVES

$$\frac{\partial G}{\partial t} = \lambda \delta^2 (1 + \alpha^2 \tilde{T}_t) -$$

$$-\frac{1}{2(T-t)\sqrt{2\pi}}[(K-w)d_2 \exp -\frac{1}{2}d_1^2 + \frac{1}{\lambda}d_1 \exp -\frac{1}{2}d_2^2]$$

SECOND DERIVATIVE I

We have

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\frac{\partial^2 \Psi}{\partial \lambda^2} - \frac{\partial \Phi}{\partial x}\frac{\partial^2 G}{\partial \lambda^2}}{(\frac{\partial G}{\partial \lambda})^2}$$

FURTHER CALCULATIONS

We recall that

$$\frac{\alpha^2\lambda^2}{2}\frac{\partial^2 G}{\partial\lambda^2} = \lambda\delta^2 - \frac{\partial G}{\partial t} - \alpha^2\lambda\frac{\partial G}{\partial\lambda}$$

and

$$\frac{\partial^2\Psi}{\partial\lambda^2} = \frac{1}{\lambda^2} - \delta^2\tilde{T}_t - \frac{1}{\lambda^2}N(d_2) + \frac{1}{\lambda^2\alpha\sqrt{2\pi(T-t)}}\exp -\frac{1}{2}d_2^2$$

NEGATIVE STATEMENT I

From the computations above, we conclude that the value function has smooth derivatives. Of course, the smoothness is only valid for $t < T$. This implies that Bellman equation is not satisfied. In addition, the generic solution of Bellman equation is different from the value function.

REIMBURSEMENT I

To apply the preceding theory to the problem of credit risk, we recall that the initial wealth x is given by

$$x = x_0 + L$$

in which L is the amount borrowed from the bank. There is a relation between L and the amount recovered by the bank at time T, which is
$\min(K, \hat{X}(T))$. So we must have (chosing a discount factor r)

$$L = \exp -r(T-t)\, E \min(K, \hat{X}(T)) \tag{42}$$

Recalling the value of $\hat{X}(T)$ (30), we obtain easily

REIMBURSEMENT II

$$L = \exp -r(T-t)\, K\, E \mathbb{1}_{\lambda Z(T) < \frac{1}{w}} = K \exp -r(T-t)\, N(-d_2(\lambda(x,t))) \tag{43}$$

From the definition of λ we get $x = G(\lambda, t)$,so we obtain the following equation

$$x_0 + K \exp -r(T-t)\, N(-d_2) = (K-w)N(-d_1) + \frac{1}{\lambda} N(-d_2) - \lambda \delta^2 \tilde{T}_t \tag{44}$$

SOLUTION I

We solve (44) in λ, for K given. We want $\lambda > 0$.Once this is done, (43) provides the value of L in terms of K. We note that both the left and right hand side of (44) are decreasing functions of λ, on R^+. The left hand side decreases from $x_0 + K \exp -r(T-t)$ to x_0 and the right hand side from $+\infty$ to $-\infty$. Since they are continuous functions of λ it is certain that the two curves cross at a point $\lambda > 0$. We cannot claim uniqueness, although for economic reasons, it is hard to justify multiple solutions. At any rate, we take the smallest possible λ, which yields to the largest possible L.

AMBIGUITY I

Suppose we do not know exactly the value of α,but we know that $\alpha_0 < \alpha < \alpha_1$, with $\alpha_0 > 0$. In that case, we change the payoff functional into

$$\inf_{\alpha} J_{x,t}(\alpha; u(.), v(.))$$

where we have emphasized the dependence in α of the payoff functional. So the problem becomes

$$\sup_{u(.).v(.)} \inf_{\alpha} J_{x,t}(\alpha; u(.), v(.))$$

AMBIGUITY II

For $x > 0$, and positive controls $u(.).v(.)$ the wealth process $X(s)$ is positive and increasing in α. Therefore, $J_{x,t}(\alpha; u(.), v(.))$ is increasing in α. Hence

$$\inf_{\alpha} J_{x,t}(\alpha; u(.), v(.)) = J_{x,t}(\alpha_0; u(.), v(.))$$

Therefore, the problem reduces to considering that $\alpha = \alpha_0$.

Thanks!